THÈSE DE DOCTORAT

Structures des classes de graphes et de leurs mineurs exclus

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Résumé

Une classe de graphes est dite close par mineur si elle est close par suppressions d'arêtes, suppressions de sommets, et contractions d'arêtes. Les classes de graphes closes par mineur jouent un rôle central en théorie des graphes grâce à leurs propriétés structurelles et algorithmiques. Dans cette thèse, nous démontrons plusieurs correspondances entre la structure d'une classe de graphes close par mineur et celle de ses mineurs exclus, c'est-à-dire des graphes minimaux parmi ceux qui ne sont pas membres de cette classe.

Dans une première partie, nous démontrons une propriété structurelle pour les classes de graphes excluant une grille de hauteur fixée en tant que mineur. Pour ce faire, nous introduisons une nouvelle famille de paramètres de graphes qui généralise la profondeur arborescente et la largeur arborescente. En conséquence, nous obtenons une généralisation du Théorème de la Grille Mineure de Robertson et Seymour.

Dans une seconde partie, nous montrons, à travers plusieurs applications, comment utiliser une notion de mineurs enracinés pour résoudre des problèmes sur les mineurs de graphes. La première de ces applications est une preuve simple pour les caractérisations des classes de graphes closes par mineurs ayant une profondeur arborescente en couche ou une largeur linéaire en couche bornée. Une deuxième application consiste en des théorèmes de Structure Produit dans des classes closes par mineurs. Enfin, nous déterminons, à un facteur linéaire près, les nombres chromatiques centrés ainsi que les nombres colorant faibles de toute classe de graphes close par mineur donnée. Dans le cas où cette classe exclut un graphe planaire, nos bornes sont optimales à un facteur constant près.

Mots-clés : théorie des graphes, mineurs de graphes, largeur arborescente, profondeur arborescente, colorations centrées

Structures of graph classes and of their excluded minors

Abstract

A class of graphs is said to be minor-closed if it is closed under the following three operations: edge deletion, vertex deletion, and edge contraction. Minor-closed classes of graphs play a central role in graph theory thanks to their numerous structural and algorithmic properties. In this thesis, we prove several connections between the structure of a minor-closed class of graphs and the structure of its excluded minors, that is the minimal graphs which are not members of this class.

In a first part, we show a structural property for classes of graphs excluding a grid of fixed height as a minor. To do so, we introduce a new family of graph parameters generalizing both treedepth and treewidth. As a consequence, we obtain a qualitative strengthening of the Grid-Minor Theorem of Robertson and Seymour for graphs excluding a rectangular grid.

In a second part, we show through multiple applications how to use a notion of rooted minors to solve problems concerning graph minors. As a first application, we provide simple proofs for characterizations of minor-closed classes of graphs having bounded layered treedepth or layered pathwidth. A second application consists of Product Structure theorems in minor-closed classes of graphs. Finally, we investigate the growth rates in minor-closed classes of graphs of weak coloring numbers and centered chromatic numbers, two families of graphs parameters characterizing classes of graphs having bounded expansions. In particular, we determine, up to a linear factor, the maximum centered chromatic numbers and weak coloring numbers of the members of a given minor-closed class of graphs. In the special case where a planar graph is excluded, our bounds are tight up to a constant factor.

Keywords: graph theory, graph minors, treewidth, treedepth, centered colorings

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Contents

CHAPTER 1

Introduction

A *graph* is a pair $G = (V, E)$ where V is a finite set and E is a set of unordered pairs of elements of V. We call the elements of V the *vertices* of G, and the elements of E the *edges* of G. We denote by $V(G)$ the vertex set V and by $E(G)$ the edge set E. When there is no ambiguity, we write uv instead of $\{u, v\}$, for every edge $\{u, v\} \in E(G)$. Here are a few examples of common graphs (see also Figure 1.1). For all positive integers n, s, t,

- the *complete graph on n vertices*, denoted by K_n, is the graph with vertex set $[n] = \{1, \ldots, n\}$ and edge set $\{\{i, j\} \mid 1 \leqslant i < j \leqslant n\}$,
- the *complete bipartite graph on $s + t$ vertices*, denoted by $K_{s,t}$, is the graph with vertex set $\{(1, 1), \ldots, (1, s)\} \cup \{(2, 1), \ldots, (2, t)\}$ and edge set $\{(1, i)(2, j) \mid i \in [s], j \in [t]\}$,
- the *path graph on n vertices*, denoted by P_n, is the graph with vertex set $\{1, \ldots, n\}$ and edge set $\{\{i, i + 1\} \mid i \in [n - 1]\}$,
- the *cycle graph on n vertices*, denoted by C_n, is the graph with vertex set $\{0, \ldots, n - 1\}$ and edge set $\{\{i, (i + 1) \bmod n\} \mid i \in [n]\}$.

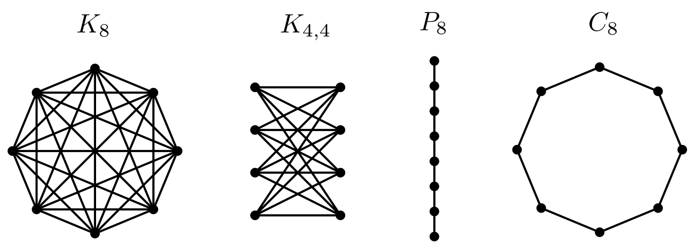


Figure 1.1: The graphs K_8, $K_{4,4}$, P_8, and C_8.

Structural Graph Theory aims at finding necessary or sufficient conditions for a graph to admit a specific structural property. A first example is the notion of forest. A *cycle* in a graph G is a sequence $(u_0, \ldots, u_{\ell-1})$ with $\ell \geqslant 3$ of pairwise distinct vertices of G such that $u_i u_{(i+1) \bmod \ell}$ is an edge in G. A *forest* is a graph that does not admit any cycle. See Figure 1.2 for an example of a forest. This definition is in terms of forbidden substructures (namely cycles), but forests also have a more structural description: the class of forests can be characterized as the smallest class of graphs satisfying

(i) the *null graph* $\emptyset = (\emptyset, \emptyset)$ is a forest, (base case)

(ii) if F is a forest, then for every $u \notin V(F)$, the graph $(V(F) \cup \{u\}, E(F))$ is a forest, and (adding an isolated vertex)

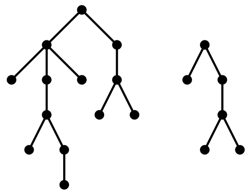


Figure 1.2: A forest.

(iii) if F is a forest, then for all $u \notin V(F)$ and $v \in V(F)$, the graph $(V(F) \cup \{u\}, E(F) \cup \{uv\})$ is a forest. (adding a leaf)

To see that, first observe that the null graph has no cycle, and that if F has no cycle, then for every $u \notin V(F)$, $(V(F) \cup \{u\}, E(F))$ has no cycle, and if $v \in V(F)$, then $(V(F) \cup \{u\}, E(F) \cup \{uv\})$ has no cycle. Hence, it remains to show that if a graph has no cycle, then can be decomposed using the rules (i), (ii), and (iii).

Before showing this, we need a few definitions. Let G be a graph. A *path* in G is a sequence $P = (u_1, \dots, u_\ell)$ of pairwise distinct vertices of G such that $u_i u_{i+1} \in E(G)$ for every $i \in [\ell - 1]$. We will often identify the path P with the graph $(\{u_1, \dots, u_\ell\}, \{u_i u_{i+1} \mid i \in [\ell - 1]\})$. The *length* of P is the integer $\ell - 1$. For all $u, v \in V(G)$, the *distance* between u and v in G, denoted by $\mathrm{dist}_G(u, v)$, is the minimum length of a path $(u_0, \dots, u_\ell)$ in G with $u_0 = u$ and $u_\ell = v$. If no such path exists, then $\mathrm{dist}_G(u, v) = +\infty$. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is $\max_{u,v \in V(G)} \mathrm{dist}_G(u, v)$. The *radius* of G, denoted by $\mathrm{diam}(G)$, is $\min_{u \in V(G)} \max_{v \in V(G)} \mathrm{dist}_G(u, v)$. For every $u \in V(G)$, the *neighborhood* of u in G, denoted by $N_G(u)$, is the set $\{v \in V(G) \mid uv \in E(G)\}$. More generally, for $U \subseteq V(G)$, we denote by $N_G(U)$ the set $\{v \in V(G) \setminus U \mid \exists u \in U, uv \in E(G)\}$. The *degree* of u in G, denoted by $d_G(u)$, is the integer $|N_G(u)|$.

The aforementioned structural characterization of forests follows from the following lemma by induction.

Lemma 1.1. *Let G be a graph. If G has no cycle, then either $V(G) = \emptyset$, or there exists in G a vertex of degree at most* 1.

Proof. Let G be a graph with no cycle and with at least one vertex. Since G has at least one vertex, say u, it admits a path, namely the sequence (u) considered as a path of length 0. Let $P = (u_1, \dots, u_\ell)$ be a path in G of maximum length. If there exists $v \in N_G(u_\ell) \setminus V(P)$, then $(u_1, \dots, u_\ell, v)$ is a path in G of larger length, contradicting the maximality of P. Hence $N_G(u_\ell) \subseteq V(P)$. If $d_G(u_\ell) \geqslant 2$, then there exists $j \in \{1, \dots, \ell - 2\}$ such that $u_j \in N_G(u_\ell)$. But then $(u_j, u_{j+1}, \dots, u_\ell)$ is a cycle in G, a contradiction. This proves $d_G(u_\ell) \leqslant 1$. □

A large part of Structural Graph Theory consists in characterizing graphs having a given structural property in terms of forbidden substructures. Here are a few examples:

- a graph is a forest if and only if it has no cycle,
- a graph is *bipartite*, that is admits a partition (A, B) of its vertex set such that every edge intersects both A and B, if and only if it has no cycle of odd length,

- a graph is perfect if and only if it has no odd hole nor odd anti-hole (Strong Perfect Graph Theorem; Chudnovsky, Robertson, Seymour, and Thomas [CRST06]). See [Die17, Section 5.5] for an introduction to perfect graphs.

Before continuing, we need to properly define the notions of "substructure" we will consider in this thesis.

Induced subgraphs. A graph H is an *induced subgraph* of a graph G if there is an injective function $\varphi\colon V(H) \to V(G)$ such that for every distinct $u, v \in V(H)$, $uv \in E(H)$ if and only if $\varphi(u)\varphi(v) \in E(G)$.

A typical example where the induced subgraph relation is relevant is when working on geometrically defined graphs. For example, the class of the *unit-disk graphs*, which are graphs of the form $(V, \{xy \mid x, y \in V, \|x - y\| \leqslant 1\})$ for some finite $V \subseteq \mathbb{R}^2$, behaves well with the induced subgraph relation: if G is a unit disk graph, then every induced subgraph of G is unit disk.

Actually, the induced subgraphs of G are, up to isomorphism, all of the form $(U, \{uv \in E(G) \mid u, v \in U\})$ for $U \subseteq V(G)$. We call the graph $(U, \{uv \in E(G) \mid u, v \in U\})$ the *subgraph of G induced by U*, and we denote it by $G[U]$. When $G[U]$ is a complete graph, we say that $G[U]$ is a *clique* in G. We also denote by $G - U$ the graph $G[V(G) \setminus U]$.

Subgraphs. A graph H is a *subgraph* of a graph G if there is an injective function $\varphi\colon V(H) \to V(G)$ such that for every $uv \in E(H)$, $\varphi(u)\varphi(v) \in E(G)$. If H is a subgraph of G, then we write $H \subseteq G$.

Alternatively, a graph H is a subgraph of G if H can be obtained from an induced subgraph of G by possibly deleting some edges. In particular, every induced subgraph of G is a subgraph of G.

Graph minors. A graph G is said to be *connected* if $V(G) \neq \emptyset$, and there is no pair A, B of nonempty disjoint subsets of $V(G)$ such that every edge of G is contained in A or in B. A graph H is a *minor* of a graph G if there is a family $(B_x \mid x \in V(H))$ of pairwise disjoint subsets of $V(G)$ such that

(m1) for every $x \in V(H)$, $G[B_x]$ is a nonempty connected subgraph of G,

(m2) for every $xy \in E(H)$, there exists $u \in B_x$ and $v \in B_y$ such that $uv \in E(G)$.

See Figure 1.3. We call such a family $(B_x \mid x \in V(H))$ a *model* of H in G. If H is not a minor of G, then we say that G is *H-minor-free*. For a family $\mathcal{H}$ of graphs, we say that G is $\mathcal{H}$-minor-free if G is H-minor-free for every $H \in \mathcal{H}$.

Note that if $(B_x \mid x \in V(H))$ is a model of H in G, and if $(C_y \mid y \in V(A))$ is a model of a graph A in H, then $(\bigcup_{x \in C_y} B_x \mid y \in V(A))$ is a model of A in G. Hence the graph minor relation is transitive.

An important operation when considering graph minors is the *edge contraction*. Let G be a graph and let $uv \in E(G)$. The graph G/uv is the graph with vertex set $G \setminus \{u, v\} \cup \{u{\star}v\}$ and edge set $(E(G) \setminus \{ww' \in E(G) \mid \{w, w'\} \cap \{u, v\} \neq \emptyset\}) \cup \{(u{\star}v, w) \mid w \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}\}$, where $u{\star}v$ denotes a fresh vertex. Less formally, G/uv is the graph obtained from G by identifying u and v. The crucial observation is that G/uv is a minor of G: for every $w \in V(G) \setminus \{u, v\}$, let $B_w = \{w\}$, and let $B_{u\star v} = \{u, v\}$. Then $(B_w \mid w \in V(G/uv))$ is a model of G/uv

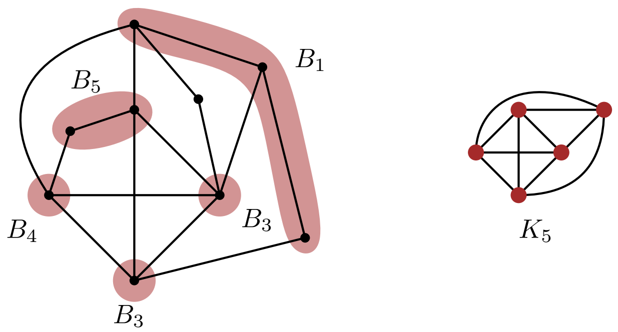


Figure 1.3: A model of K_5.

in G. To complete our basic operations, for every $u \in V(G)$, we denote by $G - u$ the graph $(V(G) \setminus \{u\}, E(G) \setminus \{uv \mid v \in N_G(u)\})$, and for every $uv \in E(G)$, we denote by $G \setminus uv$ the graph $(V(G), E(G) \setminus \{uv\})$. The graph minor relation can be then defined inductively by the fact that the graph minor relation is transitive, and that for every graph G,

(i) G is a minor of G, (reflexivity)

(ii) for every $u \in V(G)$, $G - u$ is a minor of G, (vertex deletion)

(iii) for every $uv \in E(G)$, $G \setminus uv$ is a minor of G, and (edge deletion)

(iv) for every $uv \in E(G)$, G/uv is a minor of G. (edge contraction)

The graph minor relation is the less restrictive of the usual graph containment relations. In particular, every subgraph of a graph G is a minor of G. This thesis is devoted to the study of several problems related to the notion of graph minors.

1.1 Graph minors

1.1.1 Minor-closed classes of graphs

A class of graphs $\mathcal{C}$ is *minor-closed* if for every $G \in \mathcal{C}$, for every minor H of G, $H \in \mathcal{C}$. When a graph X does not belong to $\mathcal{C}$, we say that $\mathcal{C}$ *excludes* X. Here are some examples of minor-closed classes of graphs.

Forests. The class of forests is minor-closed. We saw earlier that a graph is a forest if and only if it does not contain any cycle. A crucial observation is that a graph G has a cycle if and only if K_3 is a minor of G. Therefore, the forests can be characterized in terms of a forbidden minor as follows.

Proposition 1.2. *A graph G is a forest if and only if K_3 is not a minor of G.*

In other words, the class of forests is exactly the class of graphs that do not contain K_3 as a minor.

Outer-planar graphs. A graph G is said to be *outer-planar* if it can be drawn on the plane without any crossing, and so that every vertex of G lies on the outer face. Again, if G is outer-planar, then every minor of G is outer-planar. There are two typical graphs which are not outer-planar: K_4 and $K_{2,3}$, and so if a graph contains K_4 or $K_{2,3}$ as a minor, then it is not outer-planar. See Figure 1.4. Actually, the reciprocal also holds.

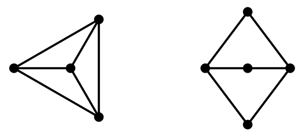


Figure 1.4: The graphs K_4 and $K_{2,3}$ are not outer-planar.

Theorem 1.3 (Folklore)**.** *A graph G is outer-planar if and only if K_4 and $K_{2,3}$ are not minors of G.*

To prove it, we will use the celebrated Menger's Theorems. Let G be a graph and let $S, T \subseteq V(G)$. An *(S,T)-path* in G is a path $(u_1, \dots, u_\ell)$ in G with $u_1 \in S$, $u_\ell \in T$, and $u_2, \dots, u_{\ell-1} \notin S \cup T$. Let $s, t \in V(G)$. An *(s,t)-path* in G is an $(\{s\}, \{t\})$-path in G. The *interior* of a path $(u_1, \dots, u_\ell)$ is the set $\{u_2, \dots, u_{\ell-1}\}$. Two paths $(u_1, \dots, u_\ell)$ and $(v_1, \dots, v_{\ell'})$ are disjoint if their vertex sets are disjoint, and *internally-disjoint* if their interiors are disjoint. In general, we say that two graphs are *disjoint* when their vertex sets are disjoint. Note that a graph G is *connected* if and only if $V(G) \neq \emptyset$, and for every pair of distinct vertices s, t of G, there is an (s,t)-path in G. The *connected components* of a graph G are the graphs of the form $G[U]$ where $U \subseteq V(G)$ is inclusion-wise maximal under the property that $G[U]$ is connected. We will use the celebrated Menger's Theorem in the following two versions.

Theorem 1.4 (Menger's Theorem 1)**.** *Let G be a graph, and let s, t be distinct nonadjacent vertices of G. For every positive integer d either*

(1) *there are d pairwise internally-disjoint (s,t)-paths in G, or*

(2) *there is a set Z of at most $d-1$ vertices in $V(G) \setminus \{s,t\}$ intersecting the interior of every (s,t)-path in G.*

Theorem 1.5 (Menger's Theorem 2)**.** *Let G be a graph, and let $S, T \subseteq V(G)$. For every positive integer d either*

(1) *there are d pairwise disjoint (S,T)-paths in G, or*

(2) *there is a set Z of at most $d-1$ vertices in G intersecting every (S,T)-path in G.*

Also, given a graph G and a set F of pairs of vertices of G, we denote by $G \cup F$ the graph $(V(G), E(G) \cup F)$.

Proof of Theorem 1.3. As mentioned earlier, it is enough to show that if a graph G does not contain K_4 and $K_{2,3}$ as a minor, then G is outer-planar. We will prove by induction on $|V(G)|$ the following stronger property.

For every graph G, if K_4 and $K_{2,3}$ are not minors of G, then there is an embedding of G in the plane such that

(i) *every vertex of G lies on the outer face, and*

(ii) *for every $uv \in E(G)$, if uv is not along the outer face, then there are three internally disjoint (u, v)-paths in G, including the length one path (u, v).*

See Figure 1.5 for an example of such an embedding.

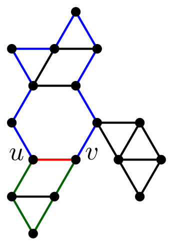


Figure 1.5: Illustration for the proof of Theorem 1.3. For every edge uv which is not along the outer face, there are three internally-disjoint (u, v)-paths in G. The edges of these paths are here depicted in red, blue, and green.

Let G be a graph that does not contain K_4 nor $K_{2,3}$ as a minor. If $|V(G)| \leqslant 3$, then the property clearly holds. Now suppose that $|V(G)| \geqslant 4$ and that the property holds for smaller graphs.

If G is a complete graph, then since K_4 is not a minor of G, we have $|V(G)| \leqslant 3$, a contradiction. Therefore G is not complete. Let $s, t \in V(G)$ distinct such that $st \notin E(G)$. Since $K_{2,3}$ is not a minor of G, there are no three pairwise internally disjoint (s,t)-paths in G. Hence, by Menger's Theorem, there exists a set Z of at most two vertices in $V(G) \setminus \{s, t\}$ intersecting the interior of every (s, t)-path in G. Assume that Z has minimum size for this property. Let A be the set of all the vertices $u \in V(G)$ such that there is an (s, u)-path in G whose interior is disjoint from Z, and let $B = V(G) \setminus (A \setminus Z)$. Let $G_1 = G[A] \cup \binom{Z}{2}$ and $G_2 = G[B] \cup \binom{Z}{2}$, where $\binom{Z}{2}$ denotes the set of all the unordered pairs of elements of Z.

First suppose that $|Z| \leqslant 1$. In particular, $\binom{Z}{2} = \emptyset$. Hence G_1 and G_2 are induced subgraphs, and so minors, of G. Therefore, K_4 and $K_{2,3}$ are not minors of G_1 and G_2. Then by applying the induction hypothesis on both G_1 and G_2, and by combining the two obtained embeddings, we obtain the desired embedding of G. Now suppose that $|Z| = 2$.

By minimality of Z, there are two $(\{s\}, Z)$-paths (resp. $(Z, \{t\})$-paths) in G whose vertex sets intersect only in s (resp. t), and whose interiors are included in A (resp. B). These two paths imply that there is a path P_1 (resp. P_2) between the two vertices in Z in $G[A]$ (resp. $G[B]$) whose interior is disjoint from Z. In particular, by contracting this path into a single edge we obtain that G_2 (resp. G_1) is a minor of G.

Let $i \in \{1, 2\}$. Since G_i is a minor of G, K_4 and $K_{2,3}$ are not minors of G. Therefore, by the induction hypothesis, there is an embedding of G_i such that every vertex lies on the outer face, and if an edge $uv \in E(G_i)$ is not along the outer face, then there are three internally disjoint (u, v)-paths in G_i. Suppose that the edge Z in G_i is not along the outer face. Then there are two internally disjoint paths in G_i between the vertices in Z which are not the edge Z itself. But then, together with P_{3-i}, this gives three internally disjoint paths in G of length at least 2 between the vertices in Z, and so $K_{2,3}$ is a minor of G, a contradiction. Hence the edge Z is along the outer face.

By gluing along the edge Z the embeddings of G_1 and G_2, we obtain an embedding of G such that every vertex lies on the outer face. Moreover, for every edge $uv \in E(G)$, if uv is not along the outer face of the resulting embedding, then one of the two following cases occurs.

Case 1. There exists $i \in \{1, 2\}$ such that $uv \in E(G_i)$ and uv is not along the outer face in the embedding of G_i. Then by hypothesis on this embedding, there are three internally disjoint (u, v)-paths in G_i. If one of them uses the edge Z, we replace in the corresponding path the edge Z by the path P_{3-i}. This gives three internally disjoint (u, v)-paths in G.

Case 2. If $\{u, v\} = Z$, then (u, v), P_1, and P_2 are three internally disjoint (u, v)-paths in G.

In both cases, the desired property holds. This proves the theorem. □

Planar graphs. Planar graphs are graphs that can be drawn on the plane without any crossing. Since every minor of a planar graph is planar, the class of planar graphs is minor-closed. The graphs K_5 and $K_{3,3}$ are not planar (see e.g. [Die17, Chapter 4]). Hence, if a graph G contains one of them as a minor, then G is not planar. See Figure 1.6. The reciprocal is known as Wagner's Theorem.

Theorem 1.6 (Wagner's Theorem)**.** *A graph G is planar if and only if K_5 and $K_{3,3}$ are not minors of G.*

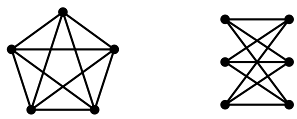


Figure 1.6: The graphs K_5 and $K_{3,3}$ are not planar.

Robertson-Seymour theorem. We saw several examples of minor-closed classes of graphs, and all of them admitted a characterization in terms of a *finite* set of forbidden minors. The celebrated Robertson-Seymour Theorem states that this is the case for every minor-closed class of graphs.

Theorem 1.7 (Robertson and Seymour [RS04])**.** *For every minor-closed class of graphs $\mathcal{C}$, there exists a finite set $\mathcal{X}$ of graphs such that for every graph G, $G \in \mathcal{C}$ if and only if for every $X \in \mathcal{X}$, X is not a minor of G.*

This result is extremely general and is of great theoretical importance. Robertson and Seymour [RS95] proved that for every fixed graph X, the problem of deciding whether X is a minor of a given n-vertex graph can be solved in time $\mathcal{O}(n^3)$. Together with Theorem 1.7, this implies that for every fixed minor-closed class of graphs $\mathcal{C}$, deciding whether a given n-vertex graph belongs to $\mathcal{C}$ can be solved in time $\mathcal{O}(n^3)$. This running time was recently improved to $n^{1+o(1)}$ by Korhonen, Pilipczuk, and Stamoulis [KPS24].

The proof of Robertson and Seymour Theorem required the introduction of numerous tools to study graph minors, and had a huge impact on both Structural and Algorithmic Graph Theory. In particular, it initiated the study of treewidth and pathwidth, two graph parameters that will play a central role in this thesis.

1.1.2 Minor-monotone graph parameters

A *graph parameter* is a function p taking as an input a graph G and returning an integer, which is invariant under isomorphism, that is for every graph G and for every bijective function $\varphi\colon V(G) \to V'$, $\mathrm{p}(G) = \mathrm{p}\left((V', \{\varphi(u)\varphi(v) \mid uv \in E(G)\})\right)$. Here are a few examples of graph parameters.

- The number of vertices $G \mapsto |V(G)|$.
- The number of edges $G \mapsto |E(G)|$.
- The maximum degree $\Delta\colon G \mapsto \max_{u \in V(G)} d_G(u)$.
- The minimum degree $\delta\colon G \mapsto \min_{u \in V(G)} d_G(u)$.

A graph parameter p is said to be *minor-monotone* if for every graph G and for every minor H of G, $\mathrm{p}(H) \leqslant \mathrm{p}(G)$. Among the aforementioned graph parameters, the number of vertices and the number of edges are minor-monotone, and the two other are not. We now present some other important minor-monotone graph parameters.

Vertex cover number. Let G be a graph. A *vertex cover* of G is a set $X \subseteq V(G)$ such that for every edge e of G, $e \cap X \neq \emptyset$. The minimum size of a vertex cover of G, denoted by $\mathrm{vc}(G)$, is the *vertex cover number* of G. The vertex cover number is minor-monotone.

Given a minor-monotone graph parameter p, a classical problem is to characterize classes of graphs $\mathcal{C}$ for which this parameter is *bounded*, that is $\max_{G \in \mathcal{C}} \mathrm{p}(G) < +\infty$. Since p is minor-monotone, p is bounded in $\mathcal{C}$ if and only if p is bounded in $\downarrow \mathcal{C} = \{H \mid \exists G \in \mathcal{C},\ H \text{ is a minor of } G\}$. As the later class is minor-closed, this reduces to the case where $\mathcal{C}$ is minor-closed.

In the case of the vertex cover number, the answer to this problem is a classical observation. Let ℓ be a positive integer. For every graph H, we denote by $\ell \cdot H$ the graph which is the union of ℓ disjoint copies of H.

Proposition 1.8. *A minor-closed class of graphs $\mathcal{C}$ has bounded vertex cover number if and only if there exists a positive integer ℓ such that $\ell \cdot K_2 \notin \mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a minor-closed class of graphs. First, $\mathrm{vc}(\ell \cdot K_2) = \ell$ for every positive integer ℓ. Hence, if $\mathcal{C}$ has bounded vertex cover number, then there exists a positive integer ℓ such that $\ell \cdot K_2 \notin \mathcal{C}$. Reciprocally, suppose that there exists a positive integer ℓ such that $\ell \cdot K_2 \notin \mathcal{C}$. Let $G \in \mathcal{C}$. Since $\mathcal{C}$ is minor-closed, $\ell \cdot K_2$ is not a minor of G. Let $e_1, \dots, e_m$ be an inclusion maximal family of pairwise disjoint edges in G. Since $\ell \cdot K_2$ is not a minor of G, we have $m \leqslant \ell - 1$. On the other hand, since $e_1, \dots, e_m$ is maximal, the set $X = \bigcup_{i \in [m]} e_i$ is a vertex cover of G, and so $\mathrm{vc}(G) \leqslant 2(\ell - 1)$. This proves that $\mathcal{C}$ has bounded vertex cover number. □

Feedback vertex set number. Let G be a graph. A *feedback vertex set* of G is a set $X \subseteq V(G)$ such that every cycle in G intersects X. In other words $G - X$ is a forest. The *feedback vertex set number* of G, denoted by $\mathrm{fvs}(G)$, is the minimum size of a feedback vertex set of G. The feedback vertex set number is minor-monotone.

Let ℓ be a positive integer. If X is a feedback vertex set of $\ell \cdot K_3$, then X intersect each of the ℓ copies of K_3. As a consequence, $\mathrm{fvs}(\ell \cdot K_3) \geqslant \ell$. See Figure 1.7. Therefore, if a graph G

contains $\ell \cdot K_3$ as a minor, then $\mathrm{fvs}(G) \geqslant \ell$. On the other hand, Erdős and Pósa [EP65] proved that if a graph does not contain $\ell \cdot K_3$ as a minor, then its feedback vertex set number is bounded by a function of ℓ.

Theorem 1.9 (Erdős-Pósa Theorem)**.** *A minor-closed class of graphs $\mathcal{C}$ has bounded feedback vertex set number if and only if there exists a positive integer ℓ such that $\ell \cdot K_3 \notin \mathcal{C}$.*

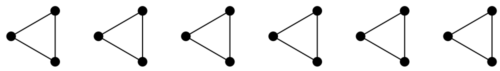


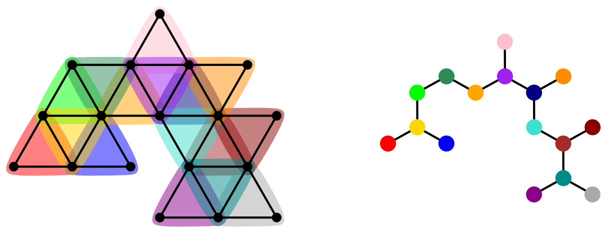
Figure 1.7: The graph $6 \cdot K_3$.

Treewidth. The treewidth is arguably the most important minor-monotone graph parameter, both for its structural and algorithmic applications. Let G be a graph. A *forest decomposition* of G is a pair $(T, (W_x \mid x \in V(T)))$, where T is a forest and $W_x \subseteq V(G)$ for every $x \in V(T)$, such that

(tw1) for every $u \in V(G)$, $\{x \in V(T) \mid u \in W_x\}$ is nonempty and induces a connected subgraph of T, and

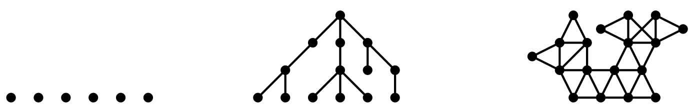
(tw2) for every $uv \in V(G)$, there exists $x \in V(T)$ such that $u, v \in W_x$.

A *tree* is a connected forest. When T is a tree, we say that $(T, (W_x \mid x \in V(T)))$ is a *tree decomposition* of G. See Figure 1.8. Most of the time, we will consider only tree decompositions. The sets W_x for $x \in V(T)$ are called the *bags* of this tree decomposition, and the set $W_x \cap W_y$ for $xy \in E(T)$ is *adhesions*. The *width* of this tree decomposition is $\max_{x \in V(T)} |W_x| - 1$, and its *adhesion* is $\max_{xy \in E(T)} |W_x \cap W_y|$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G. See Figure 1.9 for examples of graphs of small treewidth.



G T

Figure 1.8: A tree decomposition of a graph G indexed by a tree T. The bag of a vertex x of T is the set of vertices of G depicted with the same color.



$\mathrm{tw} = 0$ $\mathrm{tw} = 1$ $\mathrm{tw} = 2$

Figure 1.9: Some graphs of small treewidth.

The treewidth is minor-monotone. Indeed, if $(T, (W_x \mid x \in V(T)))$ is a tree decomposition of a graph G of minimum width, and if $(B_y \mid y \in V(H))$ is a model of a graph H in G, then $(T, (\{y \in V(H) \mid B_y \cap W_x \neq \emptyset\} \mid x \in V(T)))$ is a tree decomposition of H of width at most $\mathrm{tw}(G)$.

For every positive integer a, b, the $a \times b$ *grid* is the graph with vertex set $[a] \times [b]$ and edges all the pairs $(i,j)(i',j')$ such that $|i - i'| + |j - j'| = 1$. See Figure 1.10. We will see in Section 1.1.3

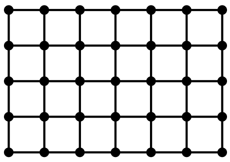


Figure 1.10: The 5×7 grid.

that for every positive integer ℓ, the $\ell \times \ell$ grid has treewidth at least ℓ. Hence, if a graph G contains the $\ell \times \ell$ grid as minor, then $\mathrm{tw}(G) \geqslant \ell$. Robertson and Seymour [RS86b] proved that the other direction also (approximately) holds. This fundamental result, known as the Grid-Minor Theorem, will be used several times in this thesis.

Theorem 1.10 (Grid-Minor Theorem [RS86b])**.** *There is a function $f_{1.10} \colon \mathbb{N} \to \mathbb{N}$ such that for every positive integer ℓ, for every graph G, if the $\ell \times \ell$ grid is not a minor of G, then*

$$\mathrm{tw}(G) < f_{1.10}(\ell).$$

As a consequence, a minor-closed class of graphs has bounded treewidth if and only if there is an integer ℓ such that the $\ell \times \ell$ grid does not belong to $\mathcal{C}$.

Pathwidth. Let G be a graph. A *path decomposition* of G is a tree decomposition $(P, (W_x \mid x \in V(P)))$ of G where P is a path graph. We will also write a path decomposition $(W_1, \dots, W_m)$ where $m = |V(P)|$, assuming that the vertices of P are $1, \dots, m$, in this order along P. The *pathwidth* of G, denoted by $\mathrm{pw}(G)$, is the minimum width of a path decomposition of G. See Figure 1.11. The same argument as for treewidth shows that pathwidth is minor-monotone.

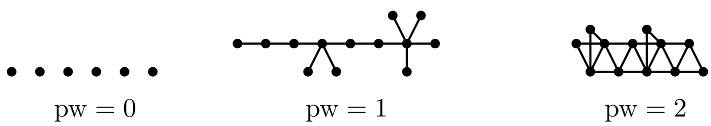


Figure 1.11: Some graphs of small pathwidth.

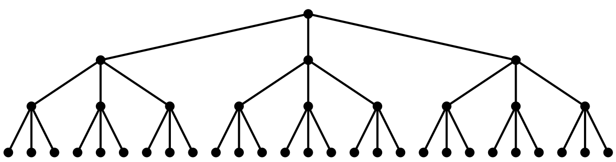


Figure 1.12: The tree T_4.

Let ℓ be a positive integer. Let T_ℓ be the complete ternary tree of depth ℓ, that is the tree with vertex set $\bigcup_{i=0}^{\ell-1}\{0,1,2\}^i$ and edge set $\bigcup_{i=0}^{\ell-2}\{\{w,wx\} \mid w \in \{0,1,2\}^i, x \in \{0,1,2\}\}$. See Figure 1.12. The tree T_ℓ has pathwidth at least $\ell-1$. The proof of this fact is by induction on ℓ. This is clear for $\ell = 0$ since $T_0 = K_1$ has pathwidth 0. Now suppose $\ell \geqslant 1$ and $\mathrm{pw}(T_{\ell-1}) \geqslant \ell-2$. Let $(W_1,\dots,W_m)$ be a path decomposition of T_ℓ of minimum width. Let Q be a (W_1,W_m)-path in T_ℓ. Then $V(Q)$ intersects W_i for every $i \in [m]$. Moreover, there exists $x \in \{0,1,2\}$ such that $U = \{x\} \times \bigcup_{i=0}^{\ell-2}\{0,1,2\}^i$ is disjoint from $V(Q)$. Therefore, $(W_1 \cap V(U),\dots,W_m \cap V(U))$ is a path decomposition of $G[U]$ of width at most $\mathrm{pw}(T_\ell)-1$. Since $G[U]$ is isomorphic to $T_{\ell-1}$, we deduce by the induction hypothesis that

$$\mathrm{pw}(T_\ell) \geqslant 1 + \mathrm{pw}(T_{\ell-1}) \geqslant \ell - 1.$$

Hence, if a graph contains T_ℓ as a minor, then $\mathrm{pw}(G) \geqslant \ell-1$. Robertson and Seymour [RS83] proved that reciprocally, if a graph does not contains a forest F as a minor, then its pathwidth is bounded by a function of F. The optimal such function was subsequently determined by Bienstock, Robertson, Seymour, and Thomas [BRST91].

Theorem 1.11 (Excluded Tree-Minor Theorem [BRST91, RS83])**.** *Let F be a forest and let G be a graph. If F is not a minor of G, then*

$$\mathrm{pw}(G) \leqslant |V(F)| - 2.$$

A short proof of this theorem was later found by Diestel [Die95]. Based on this work, we prove a qualitative strengthening of this theorem in Chapter 4, Section 4.3. As a consequence of Theorem 1.11, a minor-closed class of graphs $\mathcal{C}$ has bounded pathwidth if and only if there is a positive integer ℓ such that $T_\ell \notin \mathcal{C}$.

Treedepth. The *treedepth* is a graph parameter denoted by $\mathrm{td}(\cdot)$, defined inductively as follows. For every graph G,

(td1) if $G = \emptyset$, then $\mathrm{td}(G) = 0$;

(td2) if $G \neq \emptyset$ and G is not connected, then $\mathrm{td}(G) = \max_C \mathrm{td}(C)$ where C ranges over all the connected components of G; and

(td3) if $G \neq \emptyset$ and G is connected, then $\mathrm{td}(G) = 1 + \min_{u \in V(G)} \mathrm{td}(G-u)$.

See Figure 1.13. A direct consequence of the definition is that treedepth is minor-monotone.

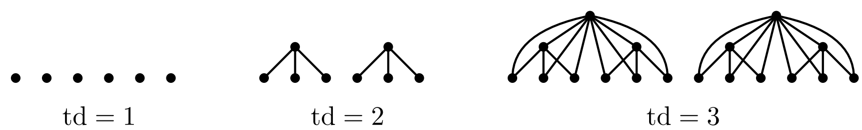


Figure 1.13: Some graphs of small treedepth.

Let k be a nonnegative integer. The path graph P_{2^k} has treedepth greater than k. Indeed, $\mathrm{td}(P_{2^0}) = 1$, and if $k > 0$, then, since P_{2^k} is connected, there exists $u \in V(P_{2^k})$ such that $\mathrm{td}(P_{2^k}) = 1 + \mathrm{td}(P_{2^k} - u)$. But since $P_{2^{k-1}}$ is a subgraph of $P_{2^k} - u$, we deduce by the induction hypothesis that $\mathrm{td}(P_{2^k}) \geqslant 1 + \mathrm{td}(P_{2^{k-1}}) > k$. Hence, if a graph contains P_{2^k} as a minor, then its treedepth is larger than k. Actually, the reciprocal also (approximately) holds.

Proposition 1.12. *Let ℓ be a positive integer and let G be a graph. If P_ℓ is not a minor of G, then*

$$\mathrm{td}(G) \leqslant \ell - 1.$$

Proof. Since $\mathrm{td}(G) = \max_C \mathrm{td}(C)$ where C ranges over all the connected components of G, we can assume that G is connected. Therefore, it is enough to prove the following.

> *Let ℓ be a positive integer, let G be a connected graph, and let $u \in V(G)$. If there is no path on ℓ vertices starting at u, then $\mathrm{td}(G) \leqslant \ell - 1$.*

We proceed by induction on ℓ. For $\ell = 1$ the result is clear since then $V(G) = \{u\}$. Now suppose $\ell > 1$ and that the result holds for $\ell - 1$. Suppose that there is no path on ℓ vertices in G starting at u. Let C be a connected component of $G - u$, and let u_C be a neighbor of u in $V(C)$. If there is a path $(u_1, \dots, u_{\ell-1})$ in C with $u_1 = u_C$, then $(u, u_1, \dots, u_{\ell-1})$ is a path on ℓ vertices in G starting at u, a contradiction. Hence, there is no such path $(u_1, \dots, u_{\ell-1})$ in C, and so by the induction hypothesis, $\mathrm{td}(C) \leqslant \ell - 2$. Then, by the definition of treedepth, $\mathrm{td}(G - u) \leqslant \ell - 2$, and finally

$$\mathrm{td}(G) \leqslant 1 + \mathrm{td}(G - u) \leqslant \ell - 1.$$ □

Hence, a minor-closed class of graphs $\mathcal{C}$ has bounded treedepth if and only if there exists a positive integer ℓ such that $P_\ell \notin \mathcal{C}$.

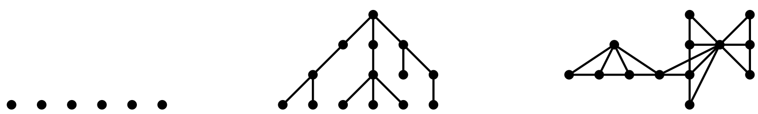
2-treedepth. The 2-treedepth is a "2-connected" variant of treedepth defined by Huynh, Joret, Micek, Seweryn, and Wollan [HJM+21]. To introduce it, we need a few definitions. A graph G is 2-*connected* if $G \neq \emptyset$ and for every $u \in V(G)$, $G - u$ is null or connected. Note that with this definition, the graphs K_1 and K_2 are 2-connected. A *block* in a graph G is a maximal 2-connected subgraph of G. The 2-*treedepth* is the graph parameter $\mathrm{td}_2(\cdot)$ defined inductively by, for every graph G,

(i) if $G = \emptyset$, then $\mathrm{td}_2(G) = 0$;

(ii) if $G \neq \emptyset$ and G is not 2-connected, then $\mathrm{td}_2(G) = \max_B \mathrm{td}_2(B)$ where B ranges over all the blocks of G; and

(iii) if G is 2-connected, then $\mathrm{td}_2(G) = 1 + \min_{u \in V(G)} \mathrm{td}_2(G - u)$.

See Figure 1.14. It is straightforward to show that 2-treedepth is minor-monotone. (See Lemma 2.5, which is a generalization of this fact.)



$\mathrm{td}_2 = 1$ $\mathrm{td}_2 = 2$ $\mathrm{td}_2 = 3$

Figure 1.14: Some graphs of small 2-treedepth. Note that every block of the right-most graph can be obtained from a tree by adding one vertex.

A standard example of graphs of large 2-treedepth are the $2 \times \ell$ grids for $\ell \geqslant 1$, which are also known as the *ladders*. We denote by L_ℓ the $2 \times \ell$ grid, for every positive integer ℓ. See Figure 1.15.

Let us prove by induction on k that $\mathrm{td}(L_{2^k}) > k$. Observe that that $\mathrm{td}_2(L_1) = 1$. Moreover, for every positive integer k, L_{2^k} is 2-connected. Hence, by the definition of 2-treedepth, there exists a vertex $u \in V(G)$ such that $\mathrm{td}_2(L_{2^k}) = 1 + \mathrm{td}_2(L_{2^k} - u)$. But since $L_{2^k} - u$ contains a subgraph isomorphic to $L_{2^{k-1}}$, we conclude that $\mathrm{td}_2(L_{2^k}) \geqslant \mathrm{td}_2(L_{2^{k-1}}) + 1$, and so $\mathrm{td}_2(L_{2^k}) > k$ by the induction hypothesis.

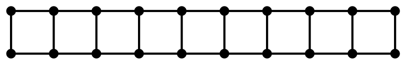


Figure 1.15: The graph L_{10}, that is the 2×10 grid.

Huynh, Joret, Micek, Seweryn, and Wollan [HJM+21] proved that reciprocally, if a graph does not contain the $2 \times \ell$ grid as minor, then its 2-treedepth is bounded by a function of ℓ.

Theorem 1.13 (Huynh, Joret, Micek, Seweryn, and Wollan [HJM+21])**.** *There is a function $f_{1.13}\colon \mathbb{N} \to \mathbb{N}$ such that for every positive integer ℓ, for every graph G, if L_ℓ is not a minor of G, then $\mathrm{td}_2(G) \leqslant f_{1.13}(\ell)$.*

In other words, a minor-closed class of graphs $\mathcal{C}$ has bounded 2-treedepth if and only if there exists a positive integer ℓ such that the $2\times\ell$ grid does not belong to $\mathcal{C}$. To conclude this presentation of these minor-monotone graph parameters, we state the inequalities between them. For every graph G,

$$\mathrm{td}(G) \leqslant \mathrm{vc}(G) + 1, \quad \mathrm{fvs}(G) \leqslant \mathrm{vc}(G), \quad \mathrm{pw}(G) \leqslant \mathrm{td}(G) - 1,$$
$$\mathrm{td}_2(G) \leqslant \mathrm{fvs}(G) + 1, \quad \mathrm{td}_2(G) \leqslant \mathrm{td}(G), \quad \mathrm{tw}(G) \leqslant \mathrm{pw}(G), \quad \mathrm{tw}(G) \leqslant \mathrm{td}_2(G) - 1.$$

Each of these inequalities is a direct consequence of the definitions, and so we omit their proofs. See Figure 1.16.

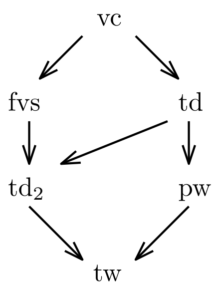


Figure 1.16: The relationships between the parameters $\mathrm{vc}, \mathrm{fvs}, \mathrm{td}, \mathrm{pw}, \mathrm{td}_2$, and tw. An arrow from a parameter p_1 to a parameter p_2 means that there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{p}_2(G) \leqslant f(\mathrm{p}_1(G))$ for every graph G.

1.1.3 Tree decompositions and treewidth

Treewidth is often described as a way to measure how a graph is close to being to a forest. Indeed, several properties of trees can be relaxed to characterize treewidth. One of the most important one is the following.

Lemma 1.14 (Helly property of subtrees)**.** *Let T be a forest and let $\mathcal{F}$ be a family of connected subgraphs of T. For every positive integer d, one of the following holds:*

(1) *there are d members $F_1, \dots, F_d$ of $\mathcal{F}$ whose vertex sets are pairwise disjoint, or*

(2) *there is a set $Z \subseteq V(T)$ of size at most $d-1$ intersecting every member of $\mathcal{F}$.*

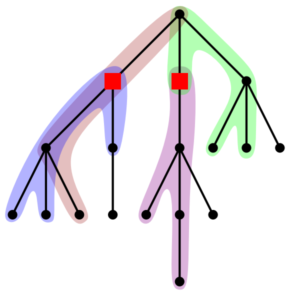


Figure 1.17: Illustration for Lemma 1.14. There are no three disjoint subtrees among the colored one. Therefore, there exists two vertices, here depicted in red, intersecting all of them.

See Figure 1.17. Lemma 1.14 has many interesting consequence. One of the simplest one is that $\mathrm{tw}(K_t) = t-1$ for every positive integer t. Taking a single bag $V(K_t)$, we obtain a tree decomposition of K_t of width $t-1$. Hence it remains to show that $\mathrm{tw}(K_t) \geqslant t-1$. This is a consequence of the following lemma.

Lemma 1.15. *Let t be a positive integer and let $(T, (W_x \mid x \in V(T)))$ be a tree decomposition of K_t. There exists $x \in V(T)$ such that $V(K_t) \subseteq W_x$.*

Proof. For every $u \in V(K_t)$, consider the connected subgraph T_u of T induced by the vertices x of T such that $u \in W_x$. Then, the subgraphs T_u for $u \in V(K_t)$ pairwise intersect by (tw2). Therefore, by Lemma 1.14 applied for $d=2$, there exists $x \in \bigcap_{u \in V(K_t)} V(T_u)$, and so $V(K_t) \subseteq W_x$. □

Before proving Lemma 1.14, we introduce the notion of rooted tree. A *rooted tree* is a tree with a marked vertex r called the *root*. In a such a rooted tree, we say that a vertex u is an *ancestor* of a vertex v, if u lies on the path from the root r to v. Then we also say that v is a *descendant* of u. Note that u is both an ancestor and descendant of itself. This gives a partial order on $V(T)$. For every vertex u which is not the root, we call the neighbor of u closest to the root the *parent* of u. If S is a subtree of T, then S inherited a root from T, which is the vertex in $V(S)$ closest to the root of T in T. For every vertex u of T, the *subtree rooted at* u is the subtree of T induced by the set of all the descendant of u in T (including u itself).

Proof of Lemma 1.14. Let d be a positive integer. We proceed by induction on d. The result is clear for $d=1$: either $\mathcal{F}$ is empty or $\mathcal{F}$ is nonempty. Now suppose $d>1$ and that the result holds for $d-1$.

We root T in an arbitrary vertex r. Among all the members of $\mathcal{F}$, let F_1 be one whose root r_1 is at maximal distance from r. Hence no descendant of r_1 distinct from r_1 is the root of a member of $\mathcal{F}$.

Let U be the set of all the descendant of u in T, let $T' = T - U$, and let $\mathcal{F}' = \{F \in \mathcal{F} \mid V(F) \cap U = \emptyset\}$. By the induction hypothesis, either

(1) there are $d-1$ members $F_2, \dots, F_d$ of $\mathcal{F}$ whose vertex sets are pairwise disjoint, or

(2) there is a set $Z' \subseteq V(T)$ of size at most $d-2$ intersecting every member of $\mathcal{F}'$.

In the first case, $F_1, F_2, \dots, F_d$ is a family of d pairwise disjoint members of $\mathcal{F}$. In the second case, we claim that $Z = Z' \cup \{r_1\}$ intersects every member of $\mathcal{F}$. Indeed, for every $F \in \mathcal{F}$, if $V(F) \cap Z' = \emptyset$, then $V(F)$ intersects U. But by the definition of F_1 and r_1, the root of F does not belong to $U \setminus \{r_1\}$. Hence $r_1 \in V(F)$ and so $V(F) \cap Z \neq \emptyset$. This proves the lemma. □

Note that Lemma 1.14 characterizes forests. Indeed, if a graph G is not a forest, then it has a cycle $(u_1, \dots, u_\ell)$, and the family $\mathcal{F} = \{G[\{u_1, u_2\}], G[\{u_2, u_3\}], G[\{u_3, \dots, u_\ell, u_1\}]\}$ has no two disjoint members but no vertex of T is in every member of $\mathcal{F}$. However, Lemma 1.14 can be extended to graphs of bounded treewidth by relaxing the second possible outcome.

Lemma 1.16. *Let t be a positive integer, let G be a graph with $\operatorname{tw}(G) < t$, and let $\mathcal{F}$ be a family of connected subgraphs of G. For every positive integer d, one of the following holds*

(1) *there are d members $F_1, \dots, F_d$ of $\mathcal{F}$ whose vertex sets are pairwise disjoint, or*

(2) *there is a set $Z \subseteq V(G)$ of size at most $t(d-1)$ intersecting every member of $\mathcal{F}$.*

We actually prove the following.

Lemma 1.17 (Statement (8.7) in [RS86b])**.** *Let t be a positive integer, let G be a graph, let $\mathcal{D}$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G. For every positive integer d, one of the following holds*

(1) *there are d members $F_1, \dots, F_d$ of $\mathcal{F}$ whose vertex sets are pairwise disjoint, or*

(2) *there is a set $Z \subseteq V(G)$ which is a union of at most $d-1$ bags of $\mathcal{D}$ such that $V(F) \cap Z \neq \emptyset$ for every $F \in \mathcal{F}$.*

Proof. Let $(T, (W_x \mid x \in V(T)))$ be a tree decomposition of G of width less than t. For every $F \in \mathcal{F}$, let $\pi(F) = T[\{x \in V(T) \mid W_x \cap V(F) \neq \emptyset\}]$. By the definition of tree decompositions, $\pi(F)$ is a connected subgraph of T for every $F \in \mathcal{F}$. Hence, by Lemma 1.14 applied to T and $\{\pi(F) \mid F \in \mathcal{F}\}$, for every positive integer d, either

(i) there are d members $F_1, \dots, F_d$ of $\mathcal{F}$ such that $\pi(F_i)$ for $i \in [d]$ are pairwise disjoint, and so $F_1, \dots, F_d$ are pairwise disjoint; or

(ii) there is a set $Z_0 \subseteq V(T)$ of size at most $d-1$ intersecting $V(\pi(F))$ for every $F \in \mathcal{F}$. Then, the set
$$Z = \bigcup\nolimits_{x \in Z_0} W_x$$
is as desired. □

A direct consequence of Lemma 1.16 is that for every positive integer ℓ, the $\ell \times \ell$ grid has treewidth at least $\ell - 1$. To see that, consider the family $\mathcal{F}$ of all the ℓ^2 subgraphs induced by sets of the form $(\{i\} \times [\ell]) \cup ([\ell] \times \{j\})$ for $i, j \in [\ell]$. Then $\mathcal{F}$ has no two disjoint members, but if a set Z of vertices intersects every member of $\mathcal{F}$, then $|Z| \geqslant \ell$. Note that a slightly more careful analysis shows that the $\ell \times \ell$ grid has treewidth ℓ.

Lemma 1.16 actually characterizes treewidth, up to a constant factor. The proof we present here is due to Bruce Reed [Ree92].

Proposition 1.18. *Let k be a positive integer and let G be a graph. If for every family $\mathcal{F}$ of connected subgraphs of G, either*

(i) *there are two disjoint members of $\mathcal{F}$, or*

(ii) *there exists $Z \subseteq V(G)$ of size at most k intersecting every member of $\mathcal{F}$;*

then

$$\mathrm{tw}(G) < 3k.$$

Proof. We will prove by induction the following stronger property.

> *Let G be a graph satisfying the hypothesis of the lemma. For every $R \subseteq V(G)$ of size at most $2k$, there exists a tree decomposition $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ of G such that*
>
> (i) *$\mathcal{D}$ has width less than $3k$, and*
>
> (ii) *there exists $r \in V(T)$ such that $R \subseteq W_r$.*

Let $R \subseteq V(G)$ of size at most $2k$. If $R = V(G)$, then the tree decomposition of G with a single bag $V(G)$ will do. Now suppose $V(G) \setminus R \neq \emptyset$, and that the result holds for instances with smaller $|V(G) \setminus R| + |V(G)|$. If $|R| < 2k$, then add any vertex to R and apply the induction hypothesis. Now assume $|R| = 2k$. Let $\mathcal{F}$ be the family of all the connected subgraphs F of G such that $|V(F) \cap R| > k$. By construction, there are no two disjoint members of $\mathcal{F}$. Hence, by hypothesis, there exists $Z \subseteq V(G)$ of size at most k such that every member of $\mathcal{F}$ intersects Z. Let $\mathcal{C}$ be the family of all the connected components of $G - Z$.

Let $C \in \mathcal{C}$, let $G_C = G[V(C) \cup N_G(V(C))]$, and let $R_C = N_G(V(C))$. Since $G_C - (Z \cap V(G_C))$ is connected and disjoint from Z, $G_C - (Z \cap V(G_C)) \notin \mathcal{F}$ and so $|V(G_C - (Z \cap V(G_C))) \cap R| \leqslant k$. Therefore, since $R_C \subseteq (V(G_C - (Z \cap V(G_C))) \cap R) \cup Z$, we have $|R_C| \leqslant 2k$. Moreover, $|V(G_C)| < |V(G)|$. Hence, by the induction hypothesis applied to G_C and R_C, there exists a tree decomposition $(T_C, (W_{C,x} \mid x \in V(T_C)))$ of G_C of width less than $3k$ and with $R_C \subseteq W_{r_C}$ for some $r_C \in V(T_C)$.

Without loss of generality, we assume that the trees T_C for $C \in \mathcal{C}$ have pairwise disjoint vertex sets. Let r be a fresh vertex, and let T be the tree defined by

$$V(T) = \{r\} \cup \bigcup_{C \in \mathcal{C}} V(T_C),$$
$$E(T) = \bigcup_{C \in \mathcal{C}} (\{rr_C\} \cup E(T_C)).$$

Then, for every $x \in V(T)$, let

$$W_x = \begin{cases} R \cup Z & \text{if } x = r, \\ W_{C,x} & \text{if } x \in V(T_C), \text{ for } C \in \mathcal{C}. \end{cases}$$

It is then straightforward to check that $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ is a tree decomposition of G. Moreover, by construction, $R \subseteq W_r$ and $\mathcal{D}$ has width less than $\max\{3k, |W_r|\} = 3k$. This proves the proposition. □

To conclude this introduction to treewidth, we define the notion of clique-sum. Let $k \in \mathbb{N}_{>0} \cup \{+\infty\}$. A $(< k)$-*clique-sum* of two graphs G_1 and G_2 is any graph G obtained from a disjoint union of G_1 and G_2 by identifying a clique K^1 in G_1 with a clique K^2 in G_2, both of the same size which is less than k, and then possibly removing some edges. See Figure 1.18. More formally, there exists two injective functions $\varphi_1\colon V(G_1) \to V(G)$ and $\varphi_2\colon V(G_2) \to V(G)$ such that (i) $E(G) \subseteq \{\varphi_1(u)\varphi_1(v) \mid uv \in E(G_1)\} \cup \{\varphi_2(u)\varphi_2(v) \mid uv \in E(G_2)\}$, and (ii) $K = \varphi_1(V(G_1)) \cap \varphi_2(V(G_2))$ has size less than k, and both $G_1[\varphi_1^{-1}(K)]$ and $G_2[\varphi_2^{-1}(K)]$ are complete. For example, a (< 1)-clique-sum is a disjoint union. When $k = +\infty$, we simply say that G is a *clique-sum* of G_1 and G_2.

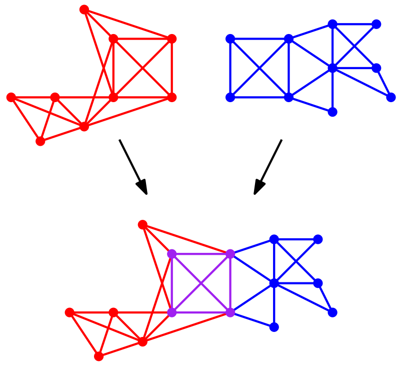


Figure 1.18: Illustration for the clique-sum operation.

The clique-sum operation is of great interest in the study of treewidth. Indeed, if G is a clique-sum of G_1 and G_2, then $\mathrm{tw}(G) \leqslant \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$. Actually, this characterizes treewidth in the following sense.

Proposition 1.19 (Halin [Hal76] [1]). *The treewidth is the largest graph parameter* tw *satisfying*

(i) $\mathrm{tw}(\emptyset) = -1$,

(ii) $\mathrm{tw}(G) \leqslant 1 + \mathrm{tw}(G - u)$ *for every graph G and every $u \in V(G)$, and*

(iii) *for all graphs G, G_1, G_2, if G is a clique-sum of G_1 and G_2, then*

$$\mathrm{tw}(G) \leqslant \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}.$$

Here, the word "largest" means that for every graph G, $\mathrm{tw}(G) = \max_{\mathrm{p}} \mathrm{p}(G)$, where p ranges over all graph parameters satisfying (i), (ii), and (iii), and that tw itself satisfies these three items. To prove it, we will use of the following notation. Let T be a tree and let x, y be two adjacent vertices of T. We denote by $T_{x|y}$ the connected component of x in $T \setminus xy$.

Proof of Proposition 1.19. Let $\mathcal{P}$ be the class of all graph parameters p satisfying (i), (ii), and (iii). We want to show that $\mathrm{tw} \in \mathcal{P}$, and that for every $\mathrm{p} \in \mathcal{P}$, for every graph G, $\mathrm{p}(G) \leqslant \mathrm{tw}(G)$.

First, we claim that $\mathrm{tw} \in \mathcal{P}$. Observe that $\mathrm{tw}(\emptyset) = -1$. Moreover, if G is a graph, $u \in V(G)$, and $(T, (W_x \mid x \in V(T)))$ is a tree decomposition of $G - u$ of minimum width, then $(T, (W_x \cup \{u\} \mid x \in V(T)))$ is a tree decomposition of G, and so $\mathrm{tw}(G) \leqslant 1 + \mathrm{tw}(G - u)$. Finally, let G be a clique-sum of two graphs G_1 and G_2. By possibly relabelling the vertices, we assume that $G = G_1 \cup G_2$, and $K = V(G_1) \cap V(G_2)$ induces a clique in both G_1 and G_2. Let

[1] Halin's articles [Hal76] and [Hal67] seem to be the first occurrences of treewidth in the litterature, before Robertson and Seymour independently rediscovered it in [RS86a].

$i \in \{1,2\}$, and let $(T_i, (W^i_x \mid x \in V(T_i)))$ be a tree decomposition of G_i of width $\mathrm{tw}(G_i)$. Since K induces a clique in G_i, by Lemma 1.15, there exists $y_i \in V(T_i)$ such that $K \subseteq W^i_{y_i}$. Without loss of generality, we assume that T_1 and T_2 have disjoint vertex sets. Let T be the tree defined by

$$V(T) = V(T_1) \cup V(T_2) \quad \text{and} \quad E(T) = E(T_1) \cup E(T_2) \cup \{y_1y_2\},$$

and for every $i \in \{1,2\}$, for every $x \in V(T_i)$, let $W_x = W^i_x$. Then, the pair $(T, (W_x \mid x \in V(T)))$ is a tree decomposition of G of width at most $\max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$. Therefore, $\mathrm{tw} \in \mathcal{P}$.

Let $\mathrm{p} \in \mathcal{P}$, let G be a graph, and let $(T, (W_x \mid x \in V(T)))$ be a tree decomposition of G of minimum width. We show by induction on $|V(T)|$ that $\mathrm{p}(G) \leqslant \mathrm{tw}(G)$. First, if $|V(T)| = 1$, then $\mathrm{tw}(G) = |V(G)| - 1 \geqslant \mathrm{p}(G)$ by (i) and (ii). Now suppose $|V(T)| \geqslant 2$ and that the result holds for smaller values of $|V(T)|$. Let $x_1x_2 \in E(T)$, and for each $i \in \{1,2\}$, let $G_i = G[\bigcup_{z \in V(T_{x_i|x_{3-i}})} W_z] \cup \binom{W_{x_1} \cap W_{x_2}}{2}$. Observe that for every $i \in \{1,2\}$, $(T_{x_i|x_{3-i}}, (W_x \mid x \in V(T_{x_i|x_{3-i}})))$ is a tree decomposition of G_i, and so $\mathrm{tw}(G_i) \leqslant \mathrm{tw}(G)$. By the induction hypothesis, this implies $\mathrm{p}(G_i) \leqslant \mathrm{tw}(G_i) \leqslant \mathrm{tw}(G)$. Moreover, since $V(G_1) \cap V(G_2)$ induces a clique in both G_1 and G_2, G is a clique-sum of G_1 and G_2. Therefore, by (iii), $\mathrm{p}(G) \leqslant \max\{\mathrm{p}(G_1), \mathrm{p}(G_2)\} \leqslant \mathrm{tw}(G)$. □

In the remaining sections of this introduction, we present the results proved in this thesis.

1.2 Excluding a rectangular grid

In a first part, we propose a family of graph parameters including treedepth, 2-treedepth, and treewidth, and we characterize in terms of forbidden minors classes of graphs in which theses parameters are bounded. Let $k \in \mathbb{N} \cup \{+\infty\}$. We define the *$k$-treedepth* as the largest graph parameter td_k satisfying

(i) $\mathrm{td}_k(\emptyset) = 0$,

(ii) $\mathrm{td}_k(G) \leqslant 1 + \mathrm{td}_k(G - u)$ for every graph G and every vertex $u \in V(G)$, and

(iii) $\mathrm{td}_k(G) \leqslant \max\{\mathrm{td}_k(G_1), \mathrm{td}_k(G_2)\}$ if G is a $(< k)$-clique-sum of G_1 and G_2, for all graphs G_1, G_2.

This gives a well-defined graph parameter because if $\mathcal{P}$ is the family of all the graph parameters satisfying (i)-(iii), then $\mathrm{td}_k \colon G \mapsto \max_{\mathrm{p} \in \mathcal{P}} \mathrm{p}(G)$ also satisfies (i)-(iii). A more explicit definition, in terms of tree decompositions, is given in Section 2.2.2. For $k = 1$, the k-treedepth coincides with treedepth, for $k = 2$, it coincides with the homonymous 2-treedepth, and for $k = +\infty$, it coincides with treewidth plus 1 by Proposition 1.19. Therefore, for every graph G,

$$\mathrm{td}(G) = \mathrm{td}_1(G) \geqslant \mathrm{td}_2(G) \geqslant \ldots \geqslant \mathrm{td}_{+\infty}(G) = \mathrm{tw}(G) + 1.$$

In Chapter 2, we characterize classes of graphs having bounded k-treedepth in terms of excluded minors.

Theorem 1.20. *Let k be a positive integer. A class $\mathcal{C}$ of graphs has bounded k-treedepth if and only if there exists an integer ℓ such that for every tree T on k vertices, no graph in $\mathcal{C}$ contains $T \square P_\ell$ as a minor.*

Here, $\square$ denotes the *Cartesian product*, defined as follows. For all graphs G_1, G_2, the graph $G_1 \square G_2$ has vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1,u_2)(v_1,v_2) \mid (u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)) \text{ or } (u_1v_1 \in E(G_1) \text{ and } u_2 = v_2)\}$. See Figure 1.19.

For $k = 1$, since K_1 is the only tree on one vertex, and because $K_1 \square P_\ell = P_\ell$ for every positive integer ℓ, we recover the fact that a minor-closed class of graphs has bounded treedepth if and only if it excludes a path. For $k = 2$, since K_2 is the only tree on two vertices, and because $K_2 \square P_\ell$ is the $2 \times \ell$ grid, we recover Huynh, Joret, Micek, Seweryn, and Wollan's [HJM+21] characterization of classes of graphs having bounded 2-treedepth (Theorem 1.13). However, for every integer k larger than 2, no such characterization was known. For example, Theorem 1.20 applied to $k = 5$ implies that for every fixed positive integer ℓ, graphs excluding $T_1 \square P_\ell, T_2 \square P_\ell$, and $T_3 \square P_\ell$ as minors have bounded 5-treedepth, where T_1, T_2, T_3 are the three trees on five vertices up to isomorphism (see Figure 1.19). This is optimal since each of the three families $\{T_1 \square P_\ell \mid \ell \geqslant 1\}, \{T_2 \square P_\ell \mid \ell \geqslant 1\}, \{T_3 \square P_\ell \mid \ell \geqslant 1\}$ has unbounded 5-treedepth.

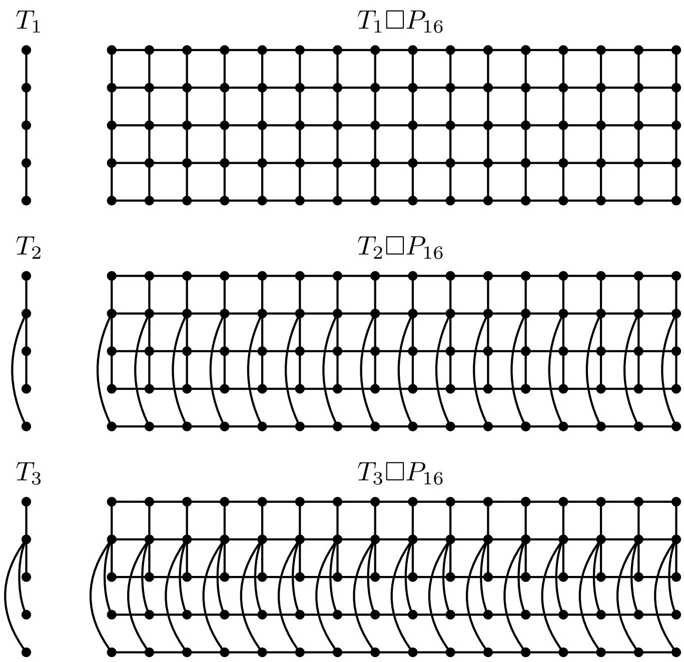


Figure 1.19: The three families of obstructions for 5-treedepth given by Theorem 1.20.

Furthermore, it is rather easy to show that for every large enough tree T and for every long enough path P, $T \square P$ contains the $\ell \times \ell$ grid as a minor. Therefore, Theorem 1.20 implies the Grid-Minor Theorem (Theorem 1.10). Actually, a more careful analysis yields the following corollary.

Corollary 1.21. *There is a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for all positive integers k, ℓ, for every graph G, if the $k \times \ell$ grid is not a minor of G, then*

$$\mathrm{td}_{2k-1}(G) \leqslant f(k, \ell).$$

Less formally, Corollary 1.21 says that in the Grid-Minor Theorem, the minimum between the height and the width of the excluded grid determines the minimum integer k such that the

k-treedepth is bounded (up to a constant factor). Note that the proof of Theorem 1.20 uses the Grid-Minor Theorem as a black box, and so this work does not give a new proof of the latter.

We also investigate the variant of k-treedepth for path decompositions, instead of tree decompositions, that we call *k-pathdepth* and denote by $\mathrm{pd}_k(\cdot)$. Informally, the *k-pathdepth* pd_k is the largest graph parameter satisfying (i) and (ii) of the definition of k-treedepth, and an analog of (iii) ensuring that the clique-sums are made in the way of a path. See Section 2.1 for a formal definition. This gives a family of parameters satisfying for every graph G

$$\mathrm{td}(G) = \mathrm{pd}_1(G) \geqslant \mathrm{pd}_2(G) \geqslant \ldots \geqslant \mathrm{pd}_{+\infty}(G) = \mathrm{pw}(G) + 1$$

and

$$\mathrm{pd}_k(G) \geqslant \mathrm{td}_k(G)$$

for every positive integer k. Hence, if a class of graphs has bounded k-pathdepth, then it has bounded k-treedepth and bounded pathwidth. Quite surprisingly, these two necessary conditions are also sufficient.

Theorem 1.22. *Let k be a positive integer and let $\mathcal{C}$ be a class of graphs. The following are equivalent.*

(1) *$\mathcal{C}$ has bounded k-pathdepth.*

(2) *$\mathcal{C}$ has bounded pathwidth and bounded k-treedepth.*

(3) *There is an integer ℓ such that for every $G \in \mathcal{C}$, $\mathrm{pw}(G) \leqslant \ell$, and for every tree T on k vertices, $T \square P_\ell$ is not a minor of G.*

Our proof actually shows the equivalence between (1) and (3), while the equivalence between (2) and (3) follows from Theorem 1.20.

1.3 Excluding a rooted minor and applications

In a second part, we investigate a notion of “rooted minors” through multiple applications. Let G be a graph and let $S \subseteq V(G)$. A model $(B_x \mid x \in V(H))$ of a graph H in G is *S-rooted* if $B_x \cap S \neq \emptyset$ for every $x \in V(H)$. Variants of this notion were investigated by many authors, see for example [RS95, Wol08, FMW13, MSW17]. Many results mentioned so far in this introduction can be extended to this setting, but most importantly, this is a rather versatile tool in the study of graph minors. In this section, we present several results obtained using this notion. See Chapter 3 for an introduction to the technique of rooted minors.

1.3.1 Layered parameters

The Grid-Minor Theorem (Theorem 1.10) characterizes classes of graphs having bounded treewidth. A natural problem arising from this result is to determine in which minor-closed classes of graphs the treewidth is *locally-bounded*, meaning that the treewidth is bounded by a function of the diameter. Eppstein [Epp00] proved that it is the case if and only if the class excludes an *apex graph*, that is a graph X which is planar or such that $X - u$ is planar for some $u \in V(X)$ (see Figure 1.20). This condition is clearly necessary since, for every positive integer ℓ, the graph obtained from the $\ell \times \ell$ grid by adding a universal vertex has treewidth at least $\ell - 1$ but diameter at most

2, while it is apex by construction. Therefore, the difficult part in the result of Eppstein [Epp00] is that for every apex graph X, X-minor-free graphs have locally-bounded treewidth. This result was later generalized by Dujmović, Morin, and Wood [DMW17] using the concept of *layered treewidth*.

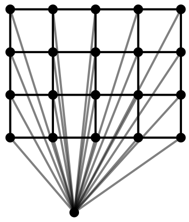


Figure 1.20: An apex graph.

A *layering* of a graph G is a sequence $L_0, L_1, \dots$ of pairwise disjoint subsets of $V(G)$ whose union is $V(G)$ such that for every edge uv of G, if $u \in L_i$ and $v \in L_j$, then $|i-j| \leqslant 1$. We call the sets L_i for $i \geqslant 0$ the *layers* of this layering. A *layered tree decomposition* of a graph G is a pair $(\mathcal{D}, \mathcal{L})$ where $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ is a tree decomposition of G, and $\mathcal{L} = (L_i \mid i \geqslant 0)$ is a layering of G. If moreover T is a path, then $(\mathcal{D}, \mathcal{L})$ is a *layered path decomposition* of G. See Figure 1.21 for an example of a layered path decomposition. The *width* of $(\mathcal{D}, \mathcal{L})$ is $\max_{x \in V(T), i \geqslant 0} |W_x \cap L_i|$. This gives a natural notion of *layered treewidth* and *layered pathwidth* as the minimum width of respectively a layered tree decomposition and a layered path decomposition of G. We denote by $\operatorname{ltw}(G)$ and $\operatorname{lpw}(G)$ respectively the layered treewidth and layered pathwidth of G.

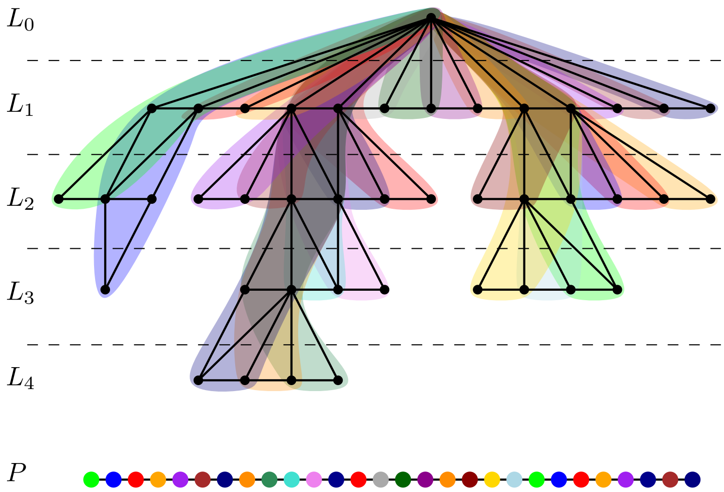


Figure 1.21: A layered path decomposition of width 2. The layering is given by the sets of vertices L_0, L_1, L_2, L_3 (with $L_i = \emptyset$ for every $i \geqslant 4$), and the bags of the path decomposition are given by the colored sets. The path P indexing this path decomposition is depicted at the bottom of the figure. It is actually a general fact that outer-planar graphs have layered pathwidth at most 2.

Since in a graph G of diameter d, the number of nonempty layers is at most $d+1$, we have $\mathrm{tw}(G) \leqslant (d+1)\,\mathrm{ltw}(G)$ and $\mathrm{pw}(G) \leqslant (d+1)\,\mathrm{lpw}(G)$, and every class of graphs having bounded layered treewidth also has locally-bounded treewidth.

Dujmović, Morin, and Wood [DMW17] proved that a minor-closed class of graphs has bounded layered treewidth if and only if it excludes an apex graph, thus generalizing Eppstein's result [Epp00].

Similarly, Dujmović, Eppstein, Joret, Morin, and Wood proved in [DEJ+20a] that a minor-closed class of graphs has bounded layered pathwidth if and only if it excludes an apex-forest, that is a graph X which is a forest, or such that $X-u$ is a forest for some $u \in V(X)$ (see Figure 1.22). This condition is necessary since there are apex-forests with large pathwidth but diameter 2, and so large layered pathwidth. In Chapter 4, we show Dujmović, Eppstein, Joret, Morin, and Wood's theorem [DEJ+20a] with a much simpler proof and an almost tight bound.

Theorem 1.23. *Let X be an apex-forest with at least two vertices. For every graph G, if G is X-minor-free, then*

$$\mathrm{lpw}(G) \leqslant 2|V(X)| - 3.$$

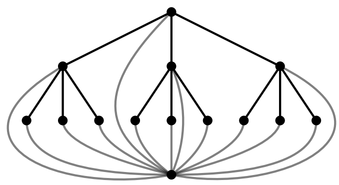


Figure 1.22: An apex-forest.

We also propose a natural counterpart for treedepth, the *layered treedepth*, denoted by $\mathrm{ltd}(\cdot)$, and we show in Chapter 4 that a minor-closed class of graphs has bounded layered treedepth if and only if it excludes a fan, that is a graph X which is a path, or such that $X-u$ is a path for some $u \in V(X)$ (see Figure 1.23). Again, this condition is clearly necessary since there are fans with diameter 2 and arbitrarily long paths, and so arbitrarily large treedepth. Therefore, the aforementioned characterization of minor-closed classes of graphs having bounded layered treedepth is implied by the following theorem.

Theorem 1.24. *For every fan X with at least three vertices, and for every graph G, if G is X-minor-free, then*

$$\mathrm{ltd}(G) \leqslant \binom{|V(X)|-1}{2}.$$

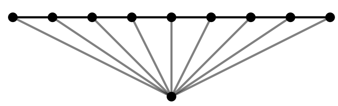


Figure 1.23: A fan.

Since in a layering of a graph G, at most $\mathrm{diam}(G)+1$ layers are nonempty, it follows that

$$\mathrm{tw}(G) \leqslant \mathrm{ltw}(G) \cdot (\mathrm{diam}(G)+1),$$
$$\mathrm{pw}(G) \leqslant \mathrm{lpw}(G) \cdot (\mathrm{diam}(G)+1),$$

and

$$\mathrm{td}(G) \leqslant \mathrm{ltd}(G) \cdot (\mathrm{diam}(G) + 1).$$

Hence, we obtain the following corollaries.

Corollary 1.25. *For every apex-forest X with at least two vertices, and for every connected graph G, if G is X-minor-free, then* $\mathrm{pw}(G) \leqslant (2|V(X)| - 3)(\mathrm{diam}(G) + 1) - 1$.

Corollary 1.26. *For every fan X with at least two vertices, and for every connected graph G, if G is X-minor-free, then* $\mathrm{td}(G) \leqslant \binom{|V(X)|-1}{2}(\mathrm{diam}(G) + 1)$.

Note that Corollaries 1.25 and 1.26 are both optimal in the following sense: there are fans of diameter 2 and arbitrarily large treedepth, and there are apex-forests of diameter 2 and arbitrarily large pathwidth.

1.3.2 Product structure

Let A, B be two graphs. The *strong product* of A and B is the graph $A \boxtimes B$ with vertex set $V(A) \times V(B)$ and edges all the pairs $(a, b)(a', b')$ such that

(i) $a = a'$ or $aa' \in E(A)$, and

(ii) $b = b'$ or $bb' \in E(B)$.

See Figure 1.24.

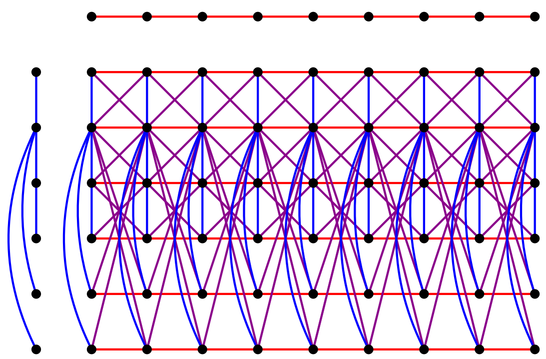


Figure 1.24: The strong product of a tree and a path.

In 2019, Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [DJM+20] proved the following structural property for planar graphs, now known as the Product Structure Theorem.

Theorem 1.27 (Product Structure Theorem [DJM+20])**.** *For every planar graph G, there exists a graph H of treewidth at most 8 such that*

$$G \subseteq H \boxtimes P$$

for some path P.

Ueckerdt, Wood, and Yi [UWY22] subsequently showed that, in this statement, H can be chosen to have treewidth at most 6. The Product Structure Theorem and its variants had a large impact on graph theory, and were used to solve many problems concerning planar graphs. Here are a few examples:

- planar graphs have bounded queue-number [DJM+20],
- planar graphs have bounded nonrepetitive chromatic number [DEJ+20b],
- planar graphs have q-centered colorings in $\mathcal{O}(q^3 \log q)$ [DMSF21],
- planar graphs have adjacency labelling schemes of size $(1 + o(1)) \log_2(n)$ [DEG+21] (see also [EJM23]),
- planar graphs have ℓ-vertex-rankings using $\mathcal{O}_\ell\left(\frac{\log n}{\log \log \log n}\right)$ colors [BDJM20],
- the clustered Hadwiger Conjecture holds [DEMW23].

In most of the applications, the following version of Theorem 1.27 is more accurate.

Theorem 1.28 (Second Product Structure Theorem [DJM+20])**.** *For every planar graph G, there exists a graph H of treewidth at most 3 such that*

$$G \subseteq H \boxtimes P \boxtimes K_3$$

for some path P.

In their original paper, Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [DJM+20] extended Theorem 1.27 to minor-closed classes of graphs that excludes an apex graph.

Theorem 1.29 (Product Structure Theorem for apex-minor-free classes [DJM+20])**.** *Let X be an apex graph. There exists a positive integer c such that, for every X-minor-free graph G, there exists a graph H of treewidth at most c such that*

$$G \subseteq H \boxtimes P$$

for some path P.

Since admitting such a product structure implies having bounded layered treewidth, this can be seen as a refinement of Dujmović, Morin, and Wood's result [DMW17] stating that minor-closed classes of graphs excluding an apex graph have bounded layered treewidth. Note that there are apex graphs with unbounded layered treewidth, and so the hypothesis X apex is necessary in such a statement.

In this context, a natural question is to what extent Theorem 1.28 can be generalized to X-minor-free graph, for an apex graph X. More precisely, given an apex graph X, what is the smallest integer integer k such that for some positive integer c, every X-minor-free graph is a subgraph of $H \boxtimes P \boxtimes K_c$ for some graph H of treewidth at most k and some path P. In Chapter 5, we answer this question by showing that $\mathrm{td}(X) - 2 \leqslant k \leqslant 2^{\mathrm{td}(X)} - 2$. The lower bound $k \geqslant \mathrm{td}(X) - 2$ follows from a construction of Ossona de Mendez, Oum, and Wood [OdMOW19] (see [DHH+24] for the details).

Theorem 1.30. *Let X be a nonnull apex graph. There exists a positive integer c such that, for every X-minor-free graph G, there exists a graph H of treewidth at most $2^{\mathrm{td}(X)} - 2$ such that*

$$G \subseteq H \boxtimes P \boxtimes K_c$$

for some path P.

We also show that when X is planar, the path factor can be removed.

Theorem 1.31. *Let X be a nonnull planar graph. There exists a positive integer c such that the following holds. For every X-minor-free graph G, there exists a graph H of treewidth at most $2^{\mathrm{td}(X)} - 2$ such that*

$$G \subseteq H \boxtimes K_c.$$

See Figure 1.25. Since for every graph H and for every positive integer c, $\mathrm{tw}(H \boxtimes K_c) \leqslant c(\mathrm{tw}(H)+1)-1$, this result can be seen as a qualitative strengthening of the Grid-Minor Theorem (Theorem 1.10).

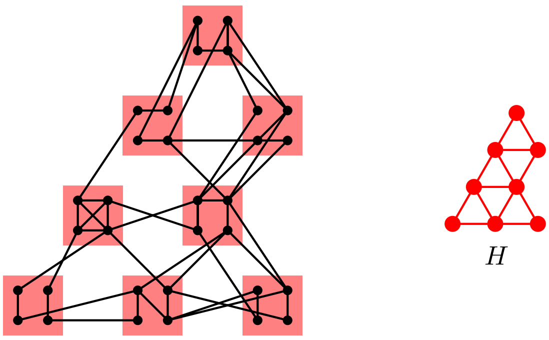


Figure 1.25: A subgraph of $H \boxtimes K_4$ where H has treewidth 2.

1.3.3 Centered colorings and weak coloring numbers

Nešetřil and Ossona de Mendez [NOdM12] introduced the concepts of bounded expansion and nowhere denseness of classes of graphs. These notions cover many well-studied classes of graphs, such as planar graphs, graphs of bounded treewidth, graphs excluding a fixed minor, graphs of bounded book-thickness, or graphs that admit drawings with a bounded number of crossings per edge as proved by Nešetřil, Ossona de Mendez, and Wood [NOdMW12]. See also the recent lecture notes of Pilipczuk, Pilipczuk, and Siebertz [PPS20]. Weak coloring numbers and centered chromatic numbers are key families of parameters capturing these concepts. In Chapter 6, we study these parameters in minor-closed classes of graphs.

Let G be a graph, let q be a positive integer, and let C be a set of colors. A coloring $\varphi\colon V(G) \to C$ of G is *q-centered* if for every connected subgraph H of G, either φ uses more than q colors on $V(H)$, or there is a color that appears exactly once on $V(H)$. The *q-centered chromatic number* of G, denoted by $\mathrm{cen}_q(G)$[2], is the least nonnegative integer k such that G admits a q-centered coloring using k colors. This is well-defined since any injective function

[2]The q-centered chromatic number is traditionally denoted by $\chi_q(\cdot)$, but we prefer here the notation $\mathrm{cen}_p(\cdot)$ for the sake of consistency with the q-th weak coloring number $\mathrm{wcol}_q(\cdot)$, and to avoid confusion with other kinds of colorings.

$\varphi\colon V(G) \to [|V(G)|]$ is a q-centered coloring of G. The following statement is one of the simplest and most important outcomes of our work.

Theorem 1.32. *Let t be an integer with $t \geqslant 2$. There exists an integer c such that, for every K_t-minor-free graph G, and for every positive integer q,*

$$\mathrm{cen}_q(G) \leqslant c \cdot q^{t-1}.$$

This improves upon the work of Pilipczuk and Siebertz [PS21] who proved that for every fixed integer t with $t \geqslant 2$, K_t-minor-free graphs have q-centered chromatic number upper bounded by a polynomial function in q. Contrary to Theorem 1.32, the degree of their polynomial is not explicitly given and arises from an application of the graph minor structure theorem by Robertson and Seymour [RS03]. On the other hand, Dębski, Micek, Schröder, and Felsner [DMSF21] showed that there exist K_t-minor-free graphs with q-centered chromatic number in $\Omega(q^{t-2})$. Hence, Theorem 1.32 is tight, up to an $\mathcal{O}(q)$ factor.

Let G be a graph, let $\Pi(G)$ be the set of all vertex orderings of G, let $\sigma \in \Pi(G)$, and let q be a nonnegative integer. For all u and v vertices of G, we say that v is *weakly q-reachable from u in* (G, σ), if there exists a path P between u and v in G of length at most q such that, for every $w \in V(P)$, $v \leqslant_\sigma w$. Let $\mathrm{WReach}_q[G, \sigma, u]$ be the set of vertices that are weakly q-reachable from u in (G, σ). The *q-th weak coloring number* of G is defined as

$$\mathrm{wcol}_q(G) = \min_{\sigma \in \Pi(G)} \max_{u \in V(G)} |\mathrm{WReach}_q[G, \sigma, u]|.$$

The state of the art for weak coloring numbers already includes a result analogous to Theorem 1.32. Namely, Van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich, and Siebertz [vdHOQ+17] showed that, for every positive integer t with $t \geqslant 2$, there exists an integer c such that for every K_t-minor-free graph G, and for every positive integer q, we have

$$\mathrm{wcol}_q(G) \leqslant c \cdot q^{t-1}.$$

In order to show the robustness of our framework, we include another family of parameters, introduced by Dvořák and Sereni [DS20], also connected to concepts of sparsity, see Dvořák [Dvo16]. For a set S, we say that $\lambda\colon \mathcal{Y} \to [0, 1]$ is a *probability distribution* on $\mathcal{Y}$ if $\sum_{Y \in \mathcal{Y}} \lambda(Y) = 1$. Let G be a graph and let q be a positive integer. The *q-th fractional treedepth-fragility rate* of G is the minimum positive integer k such that there exists a family $\mathcal{Y}$ of subsets of $V(G)$ such that

(i) $\mathrm{td}(G - Y) \leqslant k$ for every $Y \in \mathcal{Y}$; and

(ii) there exists a probability distribution λ on $\mathcal{Y}$ such that for every $u \in V(G)$, we have $\sum_{u \in Y \in \mathcal{Y}} \lambda(Y) \leqslant \frac{1}{q}$.

We denote the q-th fractional treedepth-fragility rate by $\mathrm{ftdfr}_q(G)$.

Theorem 1.33. *Let t be an integer with $t \geqslant 2$. There exists an integer c such that, for every K_t-minor-free graph G and for every positive integer q,*

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1}.$$

Dvořák and Sereni [DS20] gave bounds on the fractional treedepth-fragility rates for planar graphs and graphs of bounded treewidth. However, we are not aware of any previously known bound for K_t-minor-free graphs.

Centered colorings and weak coloring numbers are crucial tools in designing parameterized algorithms in classes of graphs of bounded expansion. For example, Pilipczuk and Siebertz [PS21] showed that if $\mathcal{C}$ is a class of graphs excluding a fixed minor, then it can be decided whether a given q-vertex graph H is a subgraph of a given n-vertex graph G in $\mathcal{C}$ in time $2^{\mathcal{O}(q \log q)} \cdot n^{\mathcal{O}(1)}$ and space $n^{\mathcal{O}(1)}$. This algorithm relies on the fact that the union of any q color classes in a q-centered coloring induces a subgraph of treedepth at most q. Therefore, finding a q-centered coloring using $q^{\mathcal{O}(1)}$ colors allows us to reduce the problem to graphs of bounded treedepth, on which the subgraph isomorphism problem can be solved efficiently. The running times of algorithms based on q-centered colorings heavily depend on the number of colors used. Weak coloring numbers characterization of sparse graphs was also used to solve algorithmic problems. Dvořák showed a constant-factor approximation for distance versions of domination number and independence number [Dvo13], with further applications in fixed-parameter algorithms and kernelization by Eickmeyer, Giannopoulou, Kreutzer, Kwon, Pilipczuk, Rabinovich, and Siebertz [EGK+17]. Grohe, Kreutzer, and Siebertz proved that deciding first-order properties is fixed-parameter tractable in nowhere dense graph classes [GKS17]. Reidl and Sullivan presented an algorithm counting the number of occurrences of a fixed induced subgraph in sparse graphs [RS23]. The time complexities of all these algorithms depend heavily on the asymptotics of wcol_q in the respective graph classes.

The growth rates of centered chromatic numbers and weak coloring numbers have been extensively studied. Grohe, Kreutzer, Rabinovich, Siebertz, and Stavropoulos [GKR+18] proved that if $\mathrm{tw}(G) \leqslant t$, then $\mathrm{wcol}_q(G) \leqslant \binom{q+t}{t}$. This is tight as for all nonnegative integers q, t they constructed a graph $G_{q,t}$ with $\mathrm{tw}(G_{q,t}) = t$ and $\mathrm{wcol}_q(G_{q,t}) = \binom{q+t}{t}$ [3]. Similarly, Pilipczuk and Siebertz [PS21] proved that if $\mathrm{tw}(G) \leqslant t$, then $\mathrm{cen}_q(G) \leqslant \binom{q+t}{t}$, which is again tight as proved by Dębski, Micek, Schröder, and Felsner [DMSF21]. Also, Dvořák and Sereni [DS20] proved that if $\mathrm{tw}(G) \leqslant t$, then $\mathrm{ftdfr}_q(G) = \mathcal{O}(q^t)$. In the class of planar graphs, Dębski, Felsner, Micek, and Schröder [DMSF21] proved that $\mathrm{cen}_q(G) = \mathcal{O}(q^3 \log q)$, van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich, and Siebertz [vdHOQ+17] proved that $\mathrm{wcol}_q(G) = \mathcal{O}(q^3)$, and Dvořák and Sereni [DS20] proved that $\mathrm{ftdfr}_q(G) = \mathcal{O}(q^3 \log q)$, while the best known lower bounds are in $\Omega(q^2 \log q)$ [DMSF21, JM22, DS20]. Interestingly, all these lower bound constructions in the planar case have bounded treewidth.

More generally, fix an arbitrary nonnull graph X and let $\mathrm{par} \in \{\mathrm{cen}, \mathrm{wcol}, \mathrm{ftdfr}\}$. What is the growth rate with respect to q of the maximum of $\mathrm{par}_q(G)$ over all X-minor-free graphs G? Most research was done for $\mathrm{par} = \mathrm{wcol}$. Van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich, and Siebertz [vdHOQ+17] showed that $\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{|V(X)|-1}\right)$. Subsequently, van den Heuvel and Wood [vdHW18] proved that $\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{\mathrm{vc}(X)+1}\right)$. Dujmović, Hickingbotham, Hodor, Joret, La, Micek, Morin, Rambaud, and Wood [DHH+24] proved that there exists an exponential function g such that $\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{g(\mathrm{td}(X))}\right)$. For $\mathrm{par} = \mathrm{cen}$, we only have $\mathrm{cen}_q(G) = \mathcal{O}(q^{h(|V(X)|)})$ for some large and non-explicit function h, see again [PS21]. For $\mathrm{par} = \mathrm{ftdfr}$, we are not aware of any polynomial bound on $\mathrm{ftdfr}_q(G)$.

[3] We recall the construction in Appendix A.1.

All of this can be seen as an effort to understand the following graph parameter. For a given nonnull graph X and $\mathrm{par} \in \{\mathrm{cen}, \mathrm{wcol}, \mathrm{ftdfr}\}$, let

$$f_{\mathrm{par}}(X) = \inf\{\alpha \in \mathbb{R} \mid \text{there exists } c > 0 \text{ such that for every } X\text{-minor-free graph } G \text{ and for every positive integer } q,\ \mathrm{par}_q(G) \leqslant c \cdot q^{\alpha}\}.$$

The question is whether f_{par} is tied to[4] some other well-established graph parameters. In simpler words, what is the property of X that governs the growth rate of $\mathrm{par}_q(G)$ for X-minor-free graphs G? Recall that for every graph X,

$$\mathrm{tw}(X) \leqslant \mathrm{pw}(X) \leqslant \mathrm{td}(X) - 1 \leqslant \mathrm{vc}(X) \leqslant |V(X)| - 1.$$

The aforementioned results imply that $\mathrm{tw}(X) - 1 \leqslant f_{\mathrm{wcol}}(X) \leqslant g(\mathrm{td}(X))$. However, f_{wcol} is not tied to any of these parameters. Indeed, neither pathwidth nor treedepth can lower bound f_{wcol}. For every positive integer k, let T_k be a complete ternary tree of vertex-height[5] k. By Robertson-Seymour Excluded Tree-Minor Theorem [RS83], there is a constant depending on k bounding the pathwidth of T_k-minor-free graphs. Also, it is easy to show that $\mathrm{wcol}_q(G) \leqslant 1 + \mathrm{pw}(G)(2q + 1)$ for every graph G[6]. Thus, $f_{\mathrm{wcol}}(T_k) \leqslant 1$ while $\mathrm{pw}(T_k) = k$ and $\mathrm{td}(T_k) = k + 1$. Next, we argue that neither treewidth nor pathwidth can upper-bound f_{wcol}. Let k be a positive integer. Recall that L_k denotes the graph $K_2 \square P_k$. There is a graph $G_{q,t}$ (constructed in [GKR⁺18]) such that $\mathrm{wcol}_q(G_{q,t}) = \Omega(q^t)$, and if $k = \Omega(2^t)$, then $G_{q,t}$ excludes L_k as a minor. Therefore, $f_{\mathrm{wcol}}(L_k) = 2^{\Omega(k)}$, and $\mathrm{tw}(L_k) \leqslant \mathrm{pw}(L_k) \leqslant 2$.

Surprisingly, the 2-treedepth is tied to each f_{par} by a linear function. Actually, we determine f_{par} up to ± 1 by introducing two 'rooted' versions of 2-treedepth. They will be called rooted 2-treedepth and simple rooted 2-treedepth, denoted respectively by $\mathrm{rtd}_2(\cdot)$ and $\mathrm{srtd}_2(\cdot)$, and they will differ only in the base cases of their definitions. We continue with the definitions of these two graph parameters.

A *linear forest* is a disjoint union of paths. A *rooted forest* is a forest where every connected component is a rooted tree. When F is a rooted forest, for every $x \in V(F)$ which is not a root, let $\mathrm{p}(F, x)$ be the parent of x in F. A forest decomposition $(F, (W_x \mid x \in V(F)))$ is *rooted* if F is a rooted forest.

Let $\mathcal{X}$ be a class of graphs. We define $\mathbf{T}(\mathcal{X})$ as the class of all the graphs G such that there is a rooted forest decomposition $(F, (W_x \mid x \in V(F)))$ of G of adhesion at most 1 such that for every $x \in V(F)$, $G[W_x \setminus W_{\mathrm{p}(F,x)}] \in \mathcal{X}$ if x is not a root, and $|W_x| \leqslant 1$ if x is a root. See Figure 1.26. Observe that $\mathcal{X} \subseteq \mathbf{T}(\mathcal{X})$. The operator $\mathbf{T}$ is monotone in the sense that for all classes of graphs $\mathcal{X}$ and $\mathcal{Y}$ with $\mathcal{X} \subseteq \mathcal{Y}$, we have $\mathbf{T}(\mathcal{X}) \subseteq \mathbf{T}(\mathcal{Y})$. We define, for every nonnegative integer t, the

[4] Two graph parameters $\mathrm{p}_1, \mathrm{p}_2$ are said to be *tied* if there are two functions α, β such that $\mathrm{p}_1(G) \leqslant \alpha(\mathrm{p}_2(G))$ and $\mathrm{p}_2(G) \leqslant \beta(\mathrm{p}_1(G))$ for every graph G.

[5] The *vertex-height* of a rooted tree T is the maximum number of vertices in a root-to-leaf path in T. If T is a non-rooted tree, then the vertex-height of T is the minimum vertex-height over every possible choice of roots.

[6] Proceed by induction on $\mathrm{pw}(G)$. We may assume that G is connected. If $\mathrm{pw}(G) = 0$, then G has no edge and so $\mathrm{wcol}_q(G) \leqslant 1$. If $\mathrm{pw}(G) > 0$, let Q be a shortest path from the first bag to the last bag of an optimal path decomposition of G. Then $\mathrm{pw}(G - V(Q)) < \mathrm{pw}(G)$ and so by induction $\mathrm{wcol}_q(G - V(Q)) \leqslant 1 + (\mathrm{pw}(G) - 1)(2q + 1)$. Let σ_0 be an ordering of $V(G)$ witnessing this fact. Now, let σ be an ordering of $V(G)$ extending σ_0 such that the vertices in $V(Q)$ appear first. Since any ball of radius q intersects a shortest path in at most $2q + 1$ vertices, it follows that σ witnesses $\mathrm{wcol}_q(G) \leqslant 1 + (\mathrm{pw}(G) - 1)(2q + 1) + (2q + 1) = 1 + \mathrm{pw}(G)(2q + 1)$.

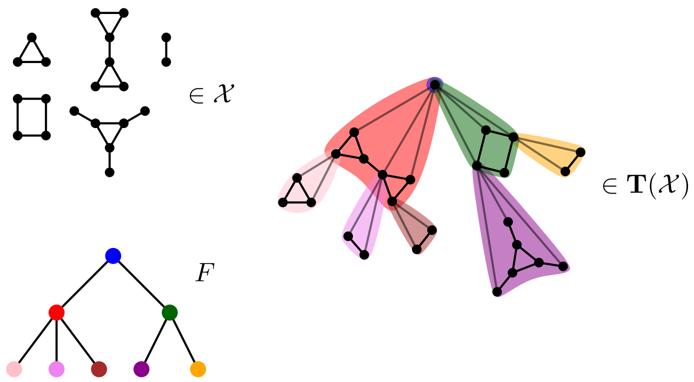


Figure 1.26: An example of a graph in $\mathbf{T}(\mathcal{X})$, and a forest decomposition witnessing this fact indexed by the tree F.

classes $\mathcal{R}_t$ and $\mathcal{S}_t$ by

$$\mathcal{R}_t = \begin{cases} \text{only the null graph} & \text{if } t = 0, \\ \text{all edgeless graphs} & \text{if } t = 1, \\ \text{all forests} & \text{if } t = 2, \\ \mathbf{T}(\mathcal{R}_{t-1}) & \text{if } t \geqslant 3, \end{cases} \quad \text{and} \quad \mathcal{S}_t = \begin{cases} \text{only the null graph} & \text{if } t = 0, \\ \text{all edgeless graphs} & \text{if } t = 1, \\ \text{all linear forests} & \text{if } t = 2, \\ \mathbf{T}(\mathcal{S}_{t-1}) & \text{if } t \geqslant 3. \end{cases}$$

See Figure 1.27 for typical examples of graphs in $\mathcal{R}_t$ and $\mathcal{S}_t$ for small values of t. In the definition of $\mathcal{R}_t$, the base cases for $t \in \{1, 2\}$ are redundant, that is, $\mathbf{T}$ applied to the class consisting only of the null graph is the class of all edgeless graphs, and $\mathbf{T}$ applied to all the edgeless graphs is the class of all forests. However, we write these base cases explicitly for a clear comparison with $\mathcal{S}_t$. It is easy to check that $\mathcal{S}_3$ contains all forests. We are ready to define the two key parameters, namely, rooted 2-treedepth (denoted by $\mathrm{rtd}_2(\cdot)$) and simple rooted 2-treedepth (denoted by $\mathrm{srtd}_2(\cdot)$). For every graph G, let[7]

$$\mathrm{rtd}_2(G) = \min\{t \in \mathbb{N} \mid G \in \mathcal{R}_t\},$$
$$\mathrm{srtd}_2(G) = \min\{t \in \mathbb{N} \mid G \in \mathcal{S}_t\}.$$

Since $\mathbf{T}$ is a monotone operator, we obtain that for every nonnegative integer t,

$$\mathcal{R}_{t+1} \supseteq \mathcal{S}_{t+1} \supseteq \mathcal{R}_t.$$

Therefore, for every graph G, we have

$$\mathrm{rtd}_2(G) \leqslant \mathrm{srtd}_2(G) \leqslant \mathrm{rtd}_2(G) + 1.$$

It is straightforward to show that moreover $\mathrm{td}_2(G) \leqslant \mathrm{rtd}_2(G) \leqslant 2\,\mathrm{td}_2(G) - 2$. See Appendix A.2, Lemma A.17. Now, we are ready to present the main contributions of Chapter 6. We give one statement for each of the three families of parameters cen, wcol, and ftdfr. However, the proofs will follow from a single abstract framework.

[7] We denote by $\mathbb{N}$ the set of the nonnegative integers and by $\mathbb{N}_{>0}$ the set of the positive integers.

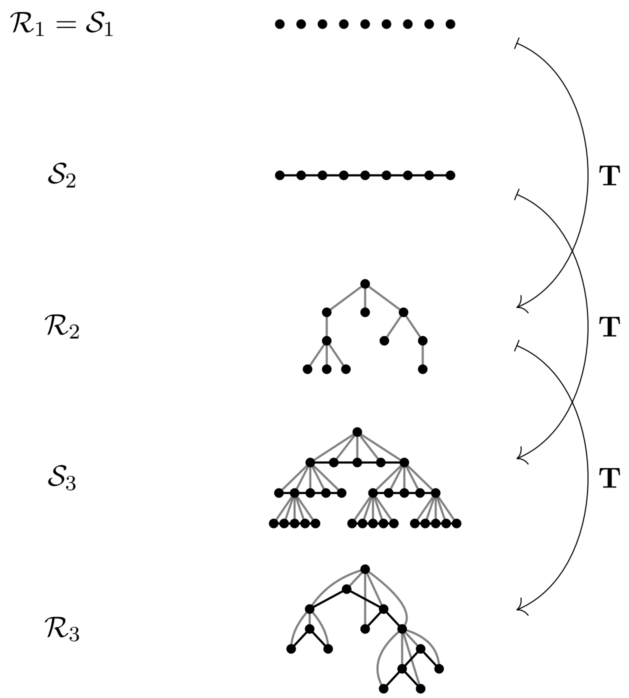


Figure 1.27: Typical graphs in $\mathcal{R}_1 = \mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{R}_2$, $\mathcal{S}_3$, and $\mathcal{R}_3$.

Theorem 1.34. *For every integer t with $t \geqslant 2$, for every graph X, there exists an integer c such that, for every X-minor-free graph G, for every integer q with $q \geqslant 2$, if $\operatorname{srtd}_2(X) \leqslant t$, then*

$$\operatorname{cen}_q(G) \leqslant c \cdot q^{t-1},$$
$$\operatorname{cen}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2};$$

and if $\operatorname{rtd}_2(X) \leqslant t$ and $t \geqslant 3$, then

$$\operatorname{cen}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\operatorname{cen}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2} \log q.$$

Theorem 1.35. *For every integer t with $t \geqslant 2$, for every graph X, there exists an integer c such that, for every X-minor-free graph G, for every integer q with $q \geqslant 2$, if $\operatorname{srtd}_2(X) \leqslant t$, then*

$$\operatorname{wcol}_q(G) \leqslant c \cdot q^{t-1},$$
$$\operatorname{wcol}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2};$$

and if $\operatorname{rtd}_2(X) \leqslant t$, then

$$\operatorname{wcol}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\operatorname{wcol}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2} \log q.$$

Theorem 1.36. *For every integer t with $t \geqslant 2$, for every graph X, there exists an integer c such that, for every X-minor-free graph G, for every integer q with $q \geqslant 2$, if $\mathrm{srtd}_2(X) \leqslant t$, then*

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1},$$
$$\mathrm{ftdfr}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2};$$

and if $\mathrm{rtd}_2(X) \leqslant t$, then

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\mathrm{ftdfr}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q.$$

Our bounds are tight up to an $\mathcal{O}(q)$ or $\mathcal{O}(\mathrm{tw}(G))$ factor. There are two setups for the lower bound constructions: a first one for X-minor-graphs when $\mathrm{rtd}_2(X) = t$, and second one for X-minor-graphs when $\mathrm{srtd}_2(X) = t$. Both of them were already present in the literature though not stated in terms of our new parameters. When $\mathrm{rtd}_2(X) = t$, this relates to a construction for graphs of bounded treewidth. This was first published in [GKR+18] where they presented a family of graphs $(G_{q,t} \mid q, t \in \mathbb{N})$ such that $\mathrm{wcol}_q(G_{q,t}) \geqslant \binom{q+t}{t}$ and the treewidth of $G_{q,t}$ at most t. The key observation is that $G_{q,t-2} \in \mathcal{R}_{t-1}$, and so $G_{q,t-2}$ is X-minor-free when $\mathrm{rtd}_2(X) = t$. When $\mathrm{srtd}_2(X) = t$, the lower bound relates to a construction given for graphs of bounded simple treewidth. This was presented first by Dębski, Felsner, Micek, and Schröder [DMSF21] to show that there are graphs G of simple treewidth at most t and with $\mathrm{cen}_q(G) = \Omega(q^{t-1} \log q)$. Both constructions are basically the same and just differ in the base case. See more details in Appendix A.1.

Our main theorems and the aforementioned lower bounds give the following general statement, which shows how to determine up to a linear factor, given the minimal excluded minors of a minor-closed class of graphs $\mathcal{C}$, the growth rates of the centered chromatic numbers, weak coloring numbers, and fractional treedepth-fragility rates in $\mathcal{C}$.

Corollary 1.37. *Let $\mathrm{par} \in \{\mathrm{cen}, \mathrm{wcol}, \mathrm{ftdfr}\}$, let $\mathcal{X}$ be a nonempty family of nonnull graphs, and let $\mathcal{C}$ be the class of all graphs G such that X is not a minor of G for all $X \in \mathcal{X}$. Let $s = \min\{\mathrm{srtd}_2(X) \mid X \in \mathcal{X}\}$ and $t = \min\{\mathrm{rtd}_2(X) \mid X \in \mathcal{X}\}$. For every integer q with $q \geqslant 2$,*

- *if $t \leqslant 1$ or $(s,t) = (2,2)$, then*

$$\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \Theta(1),$$

- *if $(s,t) = (2,3)$, then*

$$\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \begin{cases} \Theta(q) & \text{if } \mathrm{par} = \mathrm{cen}, \\ \Theta(\log q) & \text{if } \mathrm{par} \in \{\mathrm{wcol}, \mathrm{ftdfr}\}, \end{cases}$$

- *if $3 \leqslant t = s$, then for every integer q with $q \geqslant 2$,*

$$\Omega(q^{t-2}) \leqslant \max_{G \in \mathcal{C}} \mathrm{par}_q(G) \leqslant \mathcal{O}(q^{t-1}),$$

  and moreover $\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \Theta(q^{t-2})$ if $\mathcal{X}$ contains a planar graph,

- *if* $3 \leqslant t = s - 1$, *then for every integer* q *with* $q \geqslant 2$,

$$\Omega(q^{t-2} \log q) \leqslant \max_{G \in \mathcal{C}} \operatorname{par}_q(G) \leqslant \mathcal{O}(q^{t-1} \log q),$$

and moreover $\max_{G \in \mathcal{C}} \operatorname{par}_q(G) = \Theta(q^{t-2} \log q)$ *if* $\mathcal{X}$ *contains a planar graph.*

See Table 1.1 for a summary when $\mathcal{X}$ contains a planar graph. Note that, in this statement, it follows from the definitions of rtd_2 and srtd_2 that $t \leqslant s \leqslant t + 1$, and so this case distinction is exhaustive. In particular, for every nonnull graph X and for each $\operatorname{par} \in \{\operatorname{cen}, \operatorname{wcol}, \operatorname{ftdfr}\}$, the value of $f_{\operatorname{par}}(X)$ is tied to $\operatorname{rtd}_2(X)$ as follows

$$\operatorname{rtd}_2(X) - 2 \leqslant f_{\operatorname{par}}(X) \leqslant \operatorname{rtd}_2(X) - 1.$$

	$(\operatorname{cen}_q \mid q \in \mathbb{N}_{>0})$	$(\operatorname{wcol}_q \mid q \in \mathbb{N}_{>0})$	$(\operatorname{ftdfr}_q \mid q \in \mathbb{N}_{>0})$
$t \leqslant 1$ or $(t, s) = (2, 2)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
$(t, s) = (2, 3)$	$\Theta(q)$	$\Theta(\log q)$	$\Theta(\log q)$
$3 \leqslant t = s$	$\Theta(q^{t-2})$	$\Theta(q^{t-2})$	$\Theta(q^{t-2})$
$3 \leqslant t = s - 1$	$\Theta(q^{t-2} \log q)$	$\Theta(q^{t-2} \log q)$	$\Theta(q^{t-2} \log q)$

Table 1.1: For each $\operatorname{par} \in \{\operatorname{cen}, \operatorname{wcol}, \operatorname{ftdfr}\}$, Corollary 1.37 gives a complete classification of the growth rates of $\operatorname{par}_q(\cdot)$ in minor-closed classes of graphs with bounded treewidth: for $\mathcal{X}$ a finite list of graphs, at least one of them being planar, the asymptotics of $\max_{G\ \mathcal{X}\text{-minor-free}} \operatorname{par}_q(G)$ is determined by $t = \min\{\operatorname{rtd}_2(X) \mid X \in \mathcal{X}\}$ and $s = \min\{\operatorname{srtd}_2(X) \mid x \in \mathcal{X}\}$. When no graph in $\mathcal{X}$ is planar, the upper bounds are multiplied by $\mathcal{O}(q)$, which gives an estimation of $\max_{G\ \mathcal{X}\text{-minor-free}} \operatorname{par}_q(G)$ within a $\mathcal{O}(q)$ factor.

There is a number of interesting corollaries coming from the general statement of Corollary 1.37. We present some of them below. See also Tables 1.2 and 1.3. First of all, we have $\operatorname{rtd}_2(G) \leqslant \operatorname{td}(G)$ for every graph G (see Appendix A.2), which yields the following.

Corollary 1.38. *For every* $\operatorname{par} \in \{\operatorname{wcol}, \operatorname{cen}, \operatorname{ftdfr}\}$, *for every positive integer* t, *for every graph* X *with* $\operatorname{td}(X) \leqslant t$, *there exists an integer* c *such that, for every* X*-minor-free graph* G, *for every integer* q *with* $q \geqslant 2$,

$$\operatorname{par}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\operatorname{par}_q(G) \leqslant c \cdot (\operatorname{tw}(G) + 1) \cdot q^{t-2} \log q.$$

Corollary 1.39. *For every* $\operatorname{par} \in \{\operatorname{wcol}, \operatorname{cen}, \operatorname{ftdfr}\}$, *for every positive integers* s, t *with* $s \geqslant 3$, *there exists an integer* c *such that, for every* $K_{s,t}$*-minor-free graph* G, *for every integer* q *with* $q \geqslant 2$,

$$\operatorname{par}_q(G) \leqslant c \cdot q^{s} \log q,$$
$$\operatorname{par}_q(G) \leqslant c \cdot (\operatorname{tw}(G) + 1) \cdot q^{s-1} \log q.$$

Since, for every nonnegative integer g, graphs of Euler genus at most g are $K_{3,2g+3}$-minor-free, we deduce the following corollary.

Corollary 1.40. *For every* $\mathrm{par} \in \{\mathrm{wcol}, \mathrm{cen}, \mathrm{ftdfr}\}$*, for all nonnegative integers* g, w*, there exists an integer* c *such that, for every graph* G *of Euler genus at most* g *and with* $\mathrm{tw}(G) \leqslant w$*, for every integer* q *with* $q \geqslant 2$*,*

$$\mathrm{par}_q(G) \leqslant c \cdot q^2 \log q.$$

Class $\mathcal{C}$	**lower bound**		**upper bound**	
planar	$\Omega(q^2 \log q)$	[DMSF21]	$\mathcal{O}(q^3 \log q)$	[DMSF21]
planar and tw $\leqslant k$	$\Omega(q^2 \log q)$	[DMSF21]	$\mathcal{O}(q^2 \log q)$	
Euler genus $\leqslant g$	$\Omega(q^2 \log q)$	[DMSF21]	$\mathcal{O}(q^3 \log q)$	[DMSF21]
Euler genus $\leqslant g$ and tw $\leqslant k$	$\Omega(q^2 \log q)$	[DMSF21]	$\mathcal{O}(q^2 \log q)$	
outer-planar	$\Omega(q \log q)$	[DMSF21]	$\mathcal{O}(q \log q)$	[DMSF21]
$K_{2,t}$-minor-free	$\Omega(q \log q)$	[DMSF21]	$\mathcal{O}(q \log q)$	
tw $\leqslant k$	$\binom{q+k}{k}$	[DMSF21]	$\binom{q+k}{k}$	[PS21]
K_t-minor-free	$\Omega(q^{t-2})$	[DMSF21]	$\mathcal{O}(q^{t-1})$	
$K_{s,t}$-minor-free	$\Omega(q^{s-1} \log q)$	[DMSF21]	$\mathcal{O}(q^s \log q)$	
$K_{s,t}$-minor-free and tw $\leqslant k$	$\Omega(q^{s-1} \log q)$	[DMSF21]	$\mathcal{O}(q^{s-1} \log q)$	

Table 1.2: The state-of-the-art lower and upper bounds on $\max_{G \in \mathcal{C}} \mathrm{cen}_q(G)$ for some minor-closed classes of graphs $\mathcal{C}$. In red are the new bounds obtained from our results. The variables g, k, s, t are fixed positive integers with $s + 3 \leqslant t \leqslant k$.

Class $\mathcal{C}$	**lower bound**		**upper bound**	
planar	$\Omega(q^2 \log q)$	[JM22]	$\mathcal{O}(q^3)$	[vdHOQ+17]
planar and tw $\leqslant k$	$\Omega(q^2 \log q)$	[JM22]	$\mathcal{O}(q^2 \log q)$	
Euler genus $\leqslant g$	$\Omega(q^2 \log q)$	[JM22]	$\mathcal{O}(q^3)$	[vdHOQ+17]
Euler genus $\leqslant g$ and tw $\leqslant k$	$\Omega(q^2 \log q)$	[JM22]	$\mathcal{O}(q^2 \log q)$	
outer-planar	$\Omega(q \log q)$	[JM22]	$\mathcal{O}(q \log q)$	[JM22]
$K_{2,t}$-minor-free	$\Omega(q \log q)$	[JM22]	$\mathcal{O}(q \log q)$	
tw $\leqslant k$	$\binom{q+k}{k}$	[GKR+18]	$\binom{q+k}{k}$	[GKR+18]
K_t-minor-free	$\Omega(q^{t-2})$	[GKR+18]	$\mathcal{O}(q^{t-1})$	[vdHOQ+17]
$K_{s,t}$-minor-free	$\Omega(q^{s-1} \log q)$	[JM22]	$\mathcal{O}(q^s \log q)$	
$K_{s,t}$-minor-free and tw $\leqslant k$	$\Omega(q^{s-1} \log q)$	[JM22]	$\mathcal{O}(q^{s-1} \log q)$	

Table 1.3: The state-of-the-art lower and upper bounds on $\max_{G \in \mathcal{C}} \mathrm{wcol}_q(G)$ for some minor-closed classes of graphs $\mathcal{C}$. In red are the new bound deduced from our results. The variables g, k, s, t are fixed positive integers with $s + 3 \leqslant t \leqslant k$.

PART I

Excluding a rectangular grid

Chapter 2

Excluding a rectangular grid

This chapter is based on [Ram24].

The main goal of this chapter is to prove Theorem 1.20.

Theorem 1.20. *Let k be a positive integer. A class $\mathcal{C}$ of graphs has bounded k-treedepth if and only if there exists an integer ℓ such that for every tree T on k vertices, no graph in $\mathcal{C}$ contains $T\square P_\ell$ as a minor.*

In Section 2.1 are presented the main ideas behind the proof of Theorem 1.20, and in particular a proof of the implication (3)⇒(1) of Theorem 1.22. In Section 2.2, we introduce some notation and basic tools to study graph minors and tree decompositions. Section 2.3 contains the proof that graphs of the form $T\square P_\ell$ have unbounded k-treedepth as $\ell \to +\infty$, for every tree T on k vertices. This constitutes the easy direction in Theorem 1.20. Sections 2.4 to 2.6 are devoted to the proof of technical lemmas used in the final proof of Theorem 1.20, which is in Section 2.7. Then, in Section 2.8, we show that every graph of the form $T\square P_L$ for some tree T on $2k-1$ vertices contains the $k \times \ell$ grid as a minor, if L is large enough compared to ℓ. Together with Theorem 1.20, this gives Corollary 1.21.

2.1 Outline of the proof

In this section, we present the main ideas behind the proof of Theorem 1.20. First, we give an alternative definition of k-treedepth of a graph G, as the minimum width (plus one) of a "k-dismantable" tree decomposition of G. Recall that a *tree decomposition* of a graph G is a pair $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ where T is a tree and $W_x \subseteq V(G)$ for every $x \in V(T)$ satisfying

(i) for every $u \in V(T)$, $\{x \in V(T) \mid u \in W_x\}$ induces a non-empty connected subtree of T, and

(ii) for every $uv \in E(G)$, there exists $x \in V(T)$ such that $u, v \in W_x$.

More generally, when no graph is specified, we call tree decomposition any pair $(T, (W_x \mid x \in V(T)))$ satisfying (i). A tree decomposition $(T, (W_x \mid x \in V(T)))$ is *k-dismantable* if one of the following holds.

(i) T has a single vertex x and $W_x = \emptyset$; or

(ii) there is a vertex $v \in V(G)$ such that $v \in W_x$ for every $x \in V(T)$, and $(T, (W_x \setminus \{v\} \mid x \in V(T)))$ is k-dismantable; or

(iii) there is an edge $xy \in E(T)$ such that $|W_x \cap W_y| < k$ and the tree decompositions $(T_{x|y}, (W_z \mid z \in V(T_{x|y})))$ and $(T_{y|x}, (W_z \mid z \in V(T_{y|x})))$ are k-dismantable.

Recall that $T_{x|y}$ denotes the connected component of x in $T \setminus xy$, for every edge xy of T. A straightforward induction shows the following observation. See Section 2.2.2 for a proof.

Observation. *Let k be a positive integer, let G be a graph, and let t be a nonnegative integer. The k-treedepth of G is 1 plus the minimum width of a k-dismantable tree decomposition of G.*

This definition of k-treedepth based on tree decompositions will be useful in particular to show that $\mathrm{td}_k(T \square P_\ell) \to +\infty$ as $\ell \to +\infty$ for every tree T on k vertices. This proof does not represent any particular difficulty, see Section 2.3.

Hence, to prove Theorem 1.20, it remains to show that there is a function $f \colon \mathbb{N}^2 \to \mathbb{N}$ such that for every graph G and every positive integer ℓ, if $T \square P_\ell$ is not a minor of G for every tree T on k vertices, then $\mathrm{td}_k(G) \leqslant f(k, \ell)$. A first ingredient is the following lemma, which shows how to extract a minor of the form $T \square P_\ell$ for some tree T on k vertices, from k disjoint paths $Q_1, \dots, Q_k$ and sufficiently many pairwise disjoint connected subgraphs each intersecting Q_i for every $i \in [k]$.

Lemma 2.1. *There exists a function $f_{2.1} \colon \mathbb{N}^2 \to \mathbb{N}$ such that the following holds. Let k be a positive integer. For every graph G, for all k disjoint paths $Q_1, \dots, Q_k$ in G, if there exist $f_{2.1}(k, \ell)$ pairwise disjoint connected subgraphs of G each of them intersecting $V(Q_i)$ for every $i \in [k]$, then there is a tree T on k vertices such that $T \square P_\ell$ is a minor of G.*

The proof of this lemma is given in Section 2.4. The fact that trees on k vertices appear in this statement, and so in Theorem 1.20, can be easily explained. The proof of Lemma 2.1 actually yields a minor H of G with vertex set $[k] \times [k^{k-2}(\ell - 1) + 1]$ such that each "row" $\{(i, 1), \dots, (i, k^{k-2}(\ell-1)+1)\}$ for $i \in [k]$ is a path in H, and each "column" $\{(1, j), \dots, (k, j)\}$ for $j \in [k^{k-2}(\ell - 1) + 1]$ induces a connected subgraph of H. Now, each of these columns contains a spanning tree. By Cayley's formula and the pigeonhole principle, at least ℓ of these columns have the same spanning tree T when projecting on the first coordinate. By ignoring the other columns, we deduce that H contains $T \square P_\ell$ as a minor.

To illustrate how Lemma 2.1 can be used, we prove Theorem 1.22. The *k-pathdepth* of a graph G, noted $\mathrm{pd}_k(G)$, is 1 plus the minimum width of a k-dismantable path decomposition of G, that is a k-dismantable tree decomposition $(T, (W_x \mid x \in V(T)))$ of G for which T is a path. Together with the following unpublished result of Robertson and Seymour [RS] (see [Erd18] for a proof), Lemma 2.1 provides a simple proof of Theorem 1.22.

Lemma 2.2 (Robertson and Seymour [RS])**.** *For every graph G, there exists a path decomposition $\mathcal{D} = (P, (W_x \mid x \in V(P)))$ of G of width at most $\mathrm{pw}(G)$ such that for every distinct $x, y \in V(P)$, either*

(1) *there are k pairwise disjoint (W_x, W_y)-paths in G, or*

(2) *there is an edge zz' in $P[x, y]$ with $|W_z \cap W_{z'}| < k$.*

Again, knowing that $\mathrm{pd}_k(T \square P_\ell) \geqslant \mathrm{td}_k(T \square P_\ell)$ and $\mathrm{td}_k(T \square P_\ell) \to +\infty$ as $\ell \to +\infty$ for every tree T on k vertices, and $\mathrm{pd}_k(G) \geqslant \mathrm{pw}(G) + 1$ for every graph G, it remains to show the following.

Theorem 2.3. *There is a function $f_{2.3} \colon \mathbb{N}^3 \to \mathbb{N}$ such that for every positive integer ℓ and every graph G, if $T \square P_\ell$ is not a minor of G for every tree T on k vertices, then*

$$\mathrm{pd}_k(G) \leqslant f_{2.3}(k, \ell, \mathrm{pw}(G)).$$

Proof. Let k, ℓ be positive integers. For every nonnegative integer p, let

$$f_{2.3}(k, \ell, p) = \begin{cases} k & \text{if } p < k, \\ f(k, \ell, p-1) + (f_{2.1}(k, \ell) - 1)(p+1) + 2(k-1) & \text{if } p \geqslant k. \end{cases}$$

We proceed by induction on $\mathrm{pw}(G)$. Suppose that $T \square P_\ell$ is not a minor of G for every tree T on k vertices. If $\mathrm{pw}(G) < k$, then every path decomposition of G of width less than k is k-dismantable, and so $\mathrm{pd}_k(G) \leqslant k = f_{2.3}(k, \ell, \mathrm{pw}(G))$. Now assume $\mathrm{pw}(G) \geqslant k$. Then, by Lemma 2.2, there exists a path decomposition $\mathcal{D} = (P, (W_x \mid x \in V(P)))$ of G of width at most $\mathrm{pw}(G)$ satisfying (1) and (2) for every distinct $x, y \in V(P)$. Let $P^1, \dots, P^m$ be the connected components of $P \setminus \{xy \in E(P) \mid |W_x \cap W_y| < k\}$, in this order along P.

Fix some $h \in [m]$, and let x_h and y_h be respectively the first and last vertex of P^h. Let A_h be the union of the at most two adhesions neighboring P^h, that is the set $\bigcup_{zz' \in E(P), z \in V(P^h), z' \notin V(P^h)} W_z \cap W_{z'}$. Note that $|A_h| \leqslant 2(k-1)$. Finally, let $G_h = G[\bigcup_{z \in V(P^h)} W_z] \cup \binom{A_h}{2}$. Since G is a $(< k)$-clique-sum of $G_1, \dots, G_m$, we want to bound $\mathrm{pd}_k(G_h)$. If $x_h = y_h$, then $|V(G_h)| = |W_{x_h}|$ and so the path decomposition of G_h consisting of a single bag has width less than $\mathrm{pw}(G) + 1 \leqslant f_{2.3}(k, \ell, \mathrm{pw}(G))$ and is k-dismantable. Now suppose that $x_h \neq y_h$. By the properties of $\mathcal{D}$, there are k pairwise vertex-disjoint paths $Q_1, \dots, Q_k$ from W_{x_h} to W_{y_h} in G. See Figure 2.1. Let $\mathcal{F}$ be the family of all the connected subgraphs H of $G_h - A_h$ such that $V(H) \cap V(Q_i) \neq \emptyset$ for every $i \in [k]$. By Lemma 2.1, there are no $f_{2.1}(k, \ell)$ pairwise disjoint members of $\mathcal{F}$. Now, by Lemma 1.17, there are $f_{2.1}(k, \ell) - 1$ bags of $\mathcal{D}$ whose union intersects every member of $\mathcal{F}$. Hence there is a set X_0^h of at most $(f_{2.1}(k, \ell) - 1)(\mathrm{pw}(G) + 1)$ vertices in

G that intersects every member of $\mathcal{F}$. Let $X_1^h = (X_0^h \cap V(G_h)) \cup A_h$. For every connected component C of $G_h - X_1^h$, $V(C)$ is disjoint from X_0^h, and so $C \notin \mathcal{F}$. This implies that one the the paths in $\{Q_1, \dots, Q_k\}$ is disjoint from $V(C)$, and so there is a path in G from W_{x_h} to W_{y_h} disjoint from $V(C)$. Since this path intersects W_z for all $z \in P^h$, $(P^h, (W_z \cap V(C) \mid z \in V(P^h)))$ is a path decomposition of C of width at most $\mathrm{pw}(G) - 1$. By induction, $\mathrm{pd}_k(C) \leqslant f_{2.3}(k, \ell, \mathrm{pw}(G) - 1)$ and so

$$\begin{aligned} \mathrm{pd}_k\left(G_h - X_1^h\right) &\leqslant \max_{C \text{ connected component of } G_h - X_1} \mathrm{pd}_k(C) \\ &\leqslant f_{2.3}(k, \ell, \mathrm{pw}(G) - 1). \end{aligned}$$

Consider now a k-dismantable path decomposition $\mathcal{D}_h = (R^h, (W_{0,x}^h \mid x \in V(R^h)))$ of $G_h - X_1^h$ of width at most $f_{2.3}(k, \ell, \mathrm{pw}(G) - 1)$. Then $(R^h, (W_{0,x}^h \cup X_1^h \mid x \in V(R^h)))$ is a k-dismantable path decomposition of G_h of width less than $f_{2.3}(k, \ell, \mathrm{pw}(G) - 1) + (f_{2.1}(k, \ell) - 1)(\mathrm{pw}(G) + 1) + 2(k - 1) = f_{2.3}(k, \ell, \mathrm{pw}(G))$ such that A_h is included in every bag. In both cases ($x_h = y_h$ and $x_h \neq y_h$), we obtained a k-dismantable path decomposition $(R^h, (W_x^h \mid x \in V(R^h)))$ of G_h whose bags have size at most $f_{2.3}(k, \ell, \mathrm{pw}(G))$, and with A_h included in every bag. Let R be the path obtained from the disjoint union of $R^1, \dots, R^m$, and adding an edge between the last vertex of R^h and the first vertex of R^{h+1}, for every $h \in [m-1]$, and let $W'_x = W_x^h$ for every $x \in V(R^h)$, for every $h \in [m]$. Then $(R, (W'_x \mid x \in V(R)))$ is a k-dismantable path decomposition of G whose bags have size at most $f_{2.3}(k, \ell, \mathrm{pw}(G))$. This proves the theorem. □

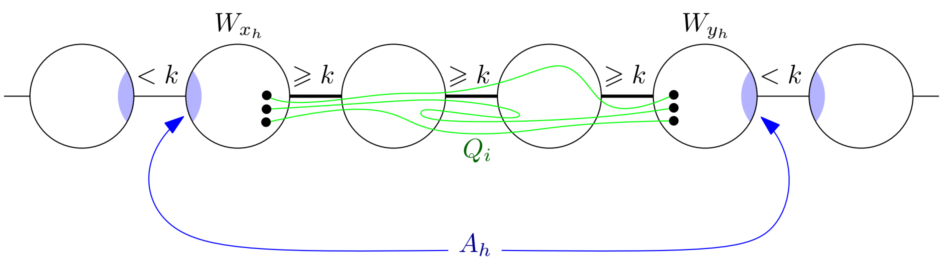


Figure 2.1: Illustration for the proof of Theorem 2.3.

This proof already contains several of the main ideas behind the proof of Theorem 1.20. The general approach to bound the k-treedepth of a graph G is to decompose G as a $(< k)$-clique-sum of graphs $G_1, \dots, G_m$ such that for every $i \in [m]$, there exists $X_i \subseteq G_i$ of bounded size such that we can bound the k-treedepth of $G_i - X_i$ (typically by induction). Hence a first step is to identify these graphs G_i. In the bounded pathwidth case, this was done using Lemma 2.2, the clique-sums being then given by the adhesions of size less than k in the obtained path decomposition. In the case $k = 2$, there is a very natural way to find these clique-sums: G is a (< 2)-clique-sum of its blocks[1], as used in [HJM+21]. For the general case, we will use a version of Lemma 2.2 for tree decompositions, which is inspired by a classical result by Thomas [Tho90] on the existence of "lean" tree decompositions. This initial tree decomposition will be built in Section 2.6. The techniques used in this section are inspired by a simple proof of Thomas' result by Bellenbaum and Diestel [BD02].

[1]The *blocks* of a graph G are the maximal subgraphs of G which consists in either a single vertex, a graph consisting of a single edge, or a 2-connected subgraph.

An issue arising from this approach when considering tree decompositions instead of path decompositions is the following. When we say that G is a $(< k)$-clique-sums of $G_1, \dots, G_m$, the graphs G_i are not necessarily minors of G, and so they may contain a graph of the form $T\square P_\ell$ for some tree T on k vertices. In the bounded pathwidth case, the graphs G_h for $h \in [m]$ had just two additional cliques, corresponding to the right and left adhesions, and so we just removed these cliques (the set A_h) to obtain a subgraph of G. This is not possible in the general case since we can have arbitrarily many such cliques. To solve this issue, we will build $G_1, \dots, G_m$ with $V(G_h) \subseteq V(G)$ for every $i \in [m]$, such that, while G_h might not be a minor of G, the "connectivity" in G_h is not higher than in G, meaning that for every $Z_1, Z_2 \subseteq V(G_h)$ both of size i, if there are i pairwise disjoint paths between Z_1 and Z_2 in G_h, then there are i pairwise disjoint paths between Z_1 and Z_2 in G. Hence, in the main proof, we will work on such an "almost minor" of G, and every time we will need the assumption that there is no minor of the form $T\square P_\ell$ for a tree T on k vertices, we will come back to the original graph using this connectivity property. This notion is developed in Section 2.5.

Another important idea in the proof of Theorem 2.3 is the induction on $\mathrm{pw}(G)$. For the general case, we want to proceed by induction on $\mathrm{tw}(G)$, which is bounded by the Grid-Minor Theorem. However, it is in practice harder to decrease the treewidth than the pathwidth. To circumvent this issue, we need a notion of "partial tree decompostion", which was already recently used in [HLMR24b] to prove some other excluded-minor results.

Let G be a graph and let $S \subseteq V(G)$. A *tree decomposition of* (G, S) is a tree decomposition $(T, (W_x \mid x \in V(T)))$ of $G[S]$ such that for every connected component C of $G - S$, there exists $x \in V(T)$ such that $N_G(V(C)) \subseteq W_x$ (the notation $N_G(V(C))$ refers to the set $\{u \in V(G) \setminus V(C) \mid \exists v \in V(C), uv \in E(G)\}$). Now, we will just assume that there is a tree decomposition of (G, S) of small width, and build by induction on this tree decomposition (actually on its adhesion) a k-dismantable tree decomposition of (G, S') of bounded width, for some $S' \subseteq V(G)$ containing S. The point of reinforcing this way the induction hypothesis is that it is much easier to decrease the width or adhesion of a tree decomposition of a subset of $V(G)$ than for the full $V(G)$. A crucial example is the following: let G be a graph, let $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ be a tree decomposition of G, let $u \in V(G)$, and let $S = \bigcup_{x \in V(T), u \in W_x} W_x \setminus \{u\}$. Now, for $T' = T[\{x \in V(T) \mid u \in W_x\}]$, $(T', (W_x \setminus \{u\} \mid x \in V(T')))$ is a tree decomposition of $(G - u, S)$ whose width (resp. adhesion) is smaller than the width (resp. adhesion) of $\mathcal{D}$. See Figure 2.2. We can now call the induction hypothesis on $(G - u, S)$, and thus decompose a superset of $N_G(u)$ in $G - u$. This idea is used in the final proof of Theorem 1.20, which is presented in Section 2.7.

2.2 Preliminaries

2.2.1 Notation

We will use the following standard notation. Let k be a positive integer and let X be a set. We denote by $[k]$ the set $\{1, \dots, k\}$, and by $\binom{X}{k}$ the family of all the subsets of X of size k.

Let G be a graph. If $U \subseteq V(G)$, we say that U is *connected in* G if there is a connected component of G containing U. If $F \subseteq \binom{V(G)}{2}$, we denote by $G \cup F$ is the graph obtained from G by adding an edge between every pair of vertices in F, and by $G \setminus F$ is the graph obtained from G by removing every edge in $F \cap E(G)$. A *path partition* of G is a sequence of $(V_0, \dots, V_\ell)$ of pairwise disjoint nonempty subsets of $V(G)$, such that $\bigcup_{0 \leqslant i \leqslant \ell} V_i = V(G)$, and for every edge

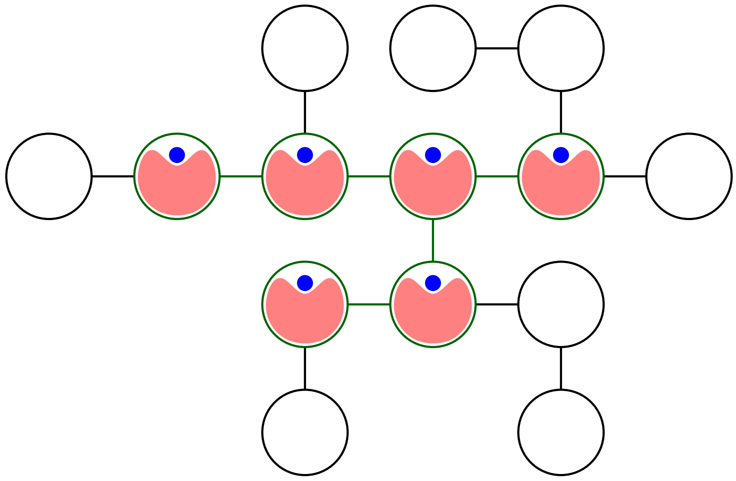


Figure 2.2: A tree decomposition of $(G-u,S)$ of width at most $\mathrm{tw}(G)-1$. The blue vertex depicts u, the tree $T' = T[\{x \in V(T) \mid u \in W_x\}]$ is depicted in green, and $S = \bigcup_{x\in V(T')} W_x \setminus \{u\}$ is the union of the red sets.

uv of G, if $i,j \in \{0,\ldots,\ell\}$ are the indices such that $u \in V_i$ and $v \in V_j$, then $|i-j| \leqslant 1$. Let $P = (p_1,\ldots,p_\ell)$ be a path in G. We denote by $\mathrm{init}(P)$ its *initial vertex* p_1, and by $\mathrm{term}(P)$ its *terminal vertex* p_ℓ. If $p_1 \in U$ and $p_\ell \in U'$ for some $U,U' \subseteq V(G)$, we say that P is a (U,U')-path, or a path from U to U'. Note that paths are thus oriented, but we will sometimes identify such a path P with the subgraph $(V(P), \{p_ip_{i+1} \mid i \in [\ell-1]\})$. A (U,U')-*cut* is a set X of vertices in G such that every (U,U')-path intersects X. Recall that Menger's Theorem asserts that the minimum size of a (U,U')-cut is equal to the maximum number of pairwise vertex-disjoint (U,U')-paths.

Let T be a tree and let $u,v \in V(T)$. We denote by $T[u,v]$ the (unique) path from u to v in T. Similarly, $T]u,v] = T[u,v]-\{u\}, T[u,v[= T[u,v]-\{v\}, T]u,v[= T[u,v]-\{u,v\}$. Moreover, recall that, if uv is an edge of T, then we denote by $T_{u|v}$ the connected component of u in $T\setminus\{uv\}$.

Let G be a graph and let S be a non empty subset of $V(G)$. A *tree decomposition of* (G,S) is a pair $\mathcal{D} = (T,(W_x \mid x \in V(T)))$ such that

(i) $\mathcal{D}$ is a tree decomposition of $G[S]$, and

(ii) for every connected component C of $G-S$, there exists $x \in V(T)$ such that $N_G(V(C)) \subseteq W_x$.

We denote by $\mathrm{tw}(G,S)$ the minimum width of a tree decomposition of (G,S).

2.2.2 k-dismantable tree decompositions

In this section, we prove some properties on k-dismantable tree decompositions. We start with the following straightforward observation.

Observation 2.4. *A graph G has k-treedepth at most t if and only if G admits a k-dismantable tree decomposition of width less than t.*

Proof. Let $\mathrm{td}'_k(G)$ be 1 plus the minimum width of a k-dismantable tree decomposition of G, for every graph G. We want to prove that $\mathrm{td}'_k(G) = \mathrm{td}_k(G)$ for every graph G. First we show that $\mathrm{td}'_k(G) \leqslant \mathrm{td}_k(G)$. It is enough to show that (i) $\mathrm{td}'_k(\emptyset) = 0$; (ii) $\mathrm{td}'_k(G) \leqslant 1+\mathrm{td}'_k(G-u)$ for every $u \in V(G)$; and (iii) $\mathrm{td}'_k(G) \leqslant \max\{\mathrm{td}'_k(G_1), \mathrm{td}'_k(G_2)\}$ if G is a $(< k)$-clique-sum of G_1 and G_2. The first point is clear. The second one comes from the fact that if $\big(T, (W_x \mid x \in V(T))\big)$ is a k-dismantable tree decomposition of $G - u$, then $\big(T, (W_x \cup \{u\} \mid x \in V(T))\big)$ is a k-dismantable tree decomposition of G. For the third point, suppose that G is obtained from the union of two graphs G_1 and G_2 with $C = V(G_1) \cap V(G_2)$ inducing a clique of size at most $k - 1$ in both G_1 and G_2. Let $\big(T_i, (W_x \mid x \in V(T_i))\big)$ is a k-dismantable tree decomposition of G_i, for $i \in \{1, 2\}$. Then, since C is a clique in G_i, the projections of $G_i[\{u\}]$ for $u \in C$ pairwise intersect. Hence by the Helly property, there exists $x_i \in V(T_i)$ intersecting all these projections, that is $C \subseteq W_{x_i}$. Now, $\big(T_1 \cup T_2 \cup \{x_1x_2\}, (W_x \mid x \in V(T_1) \cup V(T_2))\big)$ is a k-dismantable tree decomposition of G. This proves that $\mathrm{td}'_k(G) \leqslant \mathrm{td}_k(G)$ for every graph G.

Now we show by induction on $|V(G)| + |V(T)|$ that $\mathrm{td}_k(G) \leqslant \mathrm{td}'_k(G)$ for every graph G. Let $\big(T, (W_x \mid x \in V(T))\big)$ be a k-dismantable tree decomposition of G of width $t - 1$ for $t = \mathrm{td}'_k(G)$. If T has a single vertex x and $W_x = \emptyset$, then $t = 0$ and $\mathrm{td}_k(G) = 0 \leqslant t$. If $v \in V(G)$ is such that $v \in W_x$ for every $x \in V(T)$, and $\big(T, (W_x \setminus \{v\} \mid x \in V(T))\big)$ is k-dismantable, then $\mathrm{td}_k(G - v) \leqslant \mathrm{td}'_k(G - v) \leqslant t - 1$ by the induction hypothesis, and so $\mathrm{td}_k(G) \leqslant 1 + \mathrm{td}_k(G - v) \leqslant t$. If there is an edge $xy \in E(T)$ such that $|W_x \cap W_y| < k$ and the tree decompositions $\big(T_{x|y}, (W_z \mid z \in V(T_{x|y}))\big)$ and $\big(T_{y|x}, (W_z \mid z \in V(T_{y|x}))\big)$ are k-dismantable, then let $G_1 = G\left[\bigcup_{z\in V(T_{x|y})} W_z\right] \cup \binom{W_x\cap W_y}{2}$ and $G_2 = G\left[\bigcup_{z\in V(T_{y|x})} W_z\right] \cup \binom{W_x\cap W_y}{2}$. Observe that $W_x \cap W_y = V(G_1) \cap V(G_2)$ induces a clique of size less than k in both G_1 and G_2, and G is a $(< k)$-clique-sum of G_1 and G_2. Moreover, the k-dismantable tree decompositions $\big(T_{x|y}, (W_z \mid z \in V(T_{x|y}))\big), \big(T_{y|x}, (W_z \mid z \in V(T_{y|x}))\big)$ witness the fact that $\mathrm{td}'_k(G_1), \mathrm{td}'_k(G_2) \leqslant t$. Hence, since by the induction hypothesis $\mathrm{td}'_k(G_1), \mathrm{td}'_k(G_2) \leqslant t$, we have $\mathrm{td}_k(G) \leqslant \max\{\mathrm{td}_k(G_1), \mathrm{td}_k(G_2)\} \leqslant t$. This proves that $\mathrm{td}_k(G) \leqslant \mathrm{td}'_k(G)$. □

We now prove that k-treedepth is monotone for the graph minor relation.

Lemma 2.5. *Let k be a positive integer and let G, H be two graphs. If H is a minor of G, then*

$$\mathrm{td}_k(H) \leqslant \mathrm{td}_k(G).$$

Proof. Given a class of graphs $\mathcal{C}$, we denote by $\mathbf{A}(\mathcal{C})$ the class of all the graphs G such that $G \in \mathcal{C}$ or there exists $u \in V(G)$ such that $G - u \in \mathcal{C}$. We claim that

$$\text{if } \mathcal{C} \text{ is minor-closed, then } \mathbf{A}(\mathcal{C}) \text{ is minor-closed.}$$

Indeed, if G is a graph and $u \in V(G)$ such that $G - u \in \mathcal{C}$, then for every minor H of G, if $(B_x \mid x \in V(H))$ is a model of H in G, then either $u \notin \bigcup_{x\in V(H)} B_x$, and so $(B_x \mid x \in V(H))$ is a model of H in $G - u$, which implies $H \in \mathcal{C} \subseteq \mathbf{A}(\mathcal{C})$; or $u \in \bigcup_{x\in V(H)} B_x$, and then $(B_x \mid x \in V(H - x_0))$ is a model of $H - x_0$ in $G - u$, where x_0 is the unique vertex of $V(H)$ such that $u \in B_{x_0}$, which implies $H - x_0 \in \mathcal{C}$ and so $H \in \mathbf{A}(\mathcal{C})$.

Let k be a positive integer. We denote by $\mathbf{S}_k(\mathcal{C})$ the class of all the graphs G such that G is a $(< k)$-clique-sum of two graphs in $\mathcal{C}$. We claim that

$$\text{if } \mathcal{C} \text{ is minor-closed, then } \mathbf{S}_k(\mathcal{C}) \text{ is minor-closed.}$$

To see that, consider a graph G which is a $(< k)$-clique-sums of two graphs G_1 and G_2 in $\mathcal{C}$. We can assume that $V(G) = V(G_1) \cup V(G_2)$, $E(G) \subseteq E(G_1) \cup E(G_2)$, and $V(G_1) \cap V(G_2)$ induces a clique of size at most $k-1$ in both G_1 and G_2. Let H be a minor of G, and let $(B_x \mid x \in V(H))$ be a model of H in G. For $i \in \{1,2\}$, let $X_i = \{x \in V(H) \mid B_x \cap V(G_1) \neq \emptyset\}$, and let $H_i = H[X_i] \cup \binom{X_1 \cap X_2}{2}$. Then, for each $i \in \{1,2\}$, $(B_x \cap V(G_i) \mid x \in X_i)$ is a model of H_i in G_i, and so H_i is a minor of G_i. In particular, $H_i \in \mathcal{C}$. Finally, H is a $(< k)$-clique-sum of H_1 and H_2, and so $H \in \mathbf{S}_k(\mathcal{C})$.

For every nonnegative integer i, let

$$\mathbf{S}_k^i(\mathcal{C}) = \begin{cases} \mathcal{C} & \text{if } i = 0 \\ \mathbf{S}_k(\mathbf{S}_k^{i-1}(\mathcal{C})) & \text{if } i > 0. \end{cases}$$

By induction on i, $\mathbf{S}_k^i(\mathcal{C})$ is minor-closed if $\mathcal{C}$ is minor-closed. Then $\left(\mathbf{S}_k^i(\mathcal{C})\right)_{i \geqslant 0}$ is an inclusion-wise monotone sequence of minor-closed classes of graphs, and so its union $\overline{\mathbf{S}}_k(\mathcal{C})$ is minor-closed.

We can now show by induction that for every nonnegative integer t, the class $\{\mathrm{td}_k \leqslant t\}$ of all the graphs of k-treedepth at most t is minor-closed. Indeed, when $t = 0$, $\{\mathrm{td}_k \leqslant 0\} = \{\emptyset\}$ is clearly minor-closed. Moreover, if $t \geqslant 1$ and $\{\mathrm{td}_k \leqslant t-1\}$ is minor-closed, then the class $\overline{\mathbf{S}}_k(\mathbf{A}(\{\mathrm{td}_k \leqslant t-1\}))$ is also minor-closed by the previous observations. To conclude, it is enough to show that

$$\{\mathrm{td}_k \leqslant t\} = \overline{\mathbf{S}}_k(\mathbf{A}(\{\mathrm{td}_k \leqslant t-1\})). \tag{2.1}$$

The inclusion $\overline{\mathbf{S}}_k(\mathbf{A}(\{\mathrm{td}_k \leqslant t-1\})) \subseteq \{\mathrm{td}_k \leqslant t\}$ follows from the definition of k-treedepth. We now prove the other inclusion. Let G be a graph of k-treedepth at most t and let $(T, (W_x \mid x \in V(T)))$ be a k-dismantable tree decomposition of G, which exists by Observation 2.4. We want to show that $G \in \overline{\mathbf{S}}_k(\mathbf{A}(\{\mathrm{td}_k \leqslant t-1\}))$. We proceed by induction on $|V(G)| + |V(T)|$. If $V(G) = \emptyset$, then the result is clear. Now assume $V(G) \neq \emptyset$. By the definition of k-dismantable tree decompositions, one of the following holds.

Case 1. There is a vertex $v \in V(G)$ such that $v \in W_x$ for every $x \in V(T)$, and $(T, (W_x \setminus \{v\} \mid x \in V(T)))$ is k-dismantable. Then by Observation 2.4, $\mathrm{td}_k(G-u) \leqslant t-1$ and so $G \in \mathbf{A}(\{\mathrm{td}_k \leqslant t-1\}) \subseteq \overline{\mathbf{S}}_k(\mathbf{A}(\{\mathrm{td}_k \leqslant t-1\}))$.

Case 2. There is an edge $xy \in E(T)$ such that $|W_x \cap W_y| < k$ and the tree decompositions $\mathcal{D}_{x|y} = (T_{x|y}, (W_z \mid z \in V(T_{x|y})))$ and $\mathcal{D}_{y|x} = (T_{y|x}, (W_z \mid z \in V(T_{y|x})))$ are k-dismantable. Let $G_{x|y}$ (resp. $G_{y|x}$) be the graph $G[\bigcup_{z \in V(T_{x|y})} W_z] \cup \binom{W_x \cap W_y}{2}$ (resp. $G[\bigcup_{z \in V(T_{y|x})} W_z] \cup \binom{W_x \cap W_y}{2}$). The k-dismantable tree decompositions $\mathcal{D}_{x|y}$ and $\mathcal{D}_{y|x}$ witness the fact that respectively $G_{x|y}$ and $G_{y|x}$ have k-treedepth at most t. By the induction hypothesis, we deduce that $G_{x|y}, G_{y|x} \in \overline{\mathbf{S}}_k(\mathbf{A}(\{\mathrm{td}_k \leqslant t-1\}))$. Since G is a $(< k)$-clique-sum of $G_{x|y}$ and $G_{y|x}$, this implies that $G \in \overline{\mathbf{S}}_k(\mathbf{A}(\{\mathrm{td}_k \leqslant t-1\}))$. This proves (2.1), and concludes the proof of the lemma. □

For every positive integer k, for every graph G and for every nonempty set $S \subseteq V(G)$, we define the *k-treedepth of* (G,S), denoted by $\mathrm{td}_k(G,S)$ the integer 1 plus the minimum width of a k-dismantable tree decomposition of (G,S).

Lemma 2.6. *Let m, t, k be positive integers. For every graph G and for every $S_1, \dots, S_m \subseteq V(G)$, if $S = \bigcup_{i \in [m]} S_i$, then*

(a) $\mathrm{tw}(G,S)+1 \leqslant \sum_{i\in[m]}(\mathrm{tw}(G,S_i)+1)$, *and*

(b) $\mathrm{td}_k(G,S) \leqslant \sum_{i\in[m]} \mathrm{td}_k(G,S_i)$.

Proof. Since for every $S \subseteq V(G)$, $\mathrm{td}_{|V(G)|}(G,S) = \mathrm{tw}(G,S)+1$, it is enough to show the second item. We proceed by induction on m. The result is clear for $m=1$. Now suppose $m>1$ and $\mathrm{td}_k(G,\bigcup_{i\in[m-1]} S_i) \leqslant \sum_{i\in[m-1]} \mathrm{td}_k(G,S_i)$. Let $\mathcal{D} = (T,(W_x \mid x \in V(T)))$ be a k-dismantable tree decomposition of $(G,\bigcup_{i\in[m-1]} S_i)$ of width less than $\sum_{i\in[m-1]} \mathrm{td}_k(G,S_i)$. For every connected component C of $G-\bigcup_{i\in[m-1]} S_i$, we have $\mathrm{td}_k(C,S_m\cap V(C)) \leqslant \mathrm{td}_k(G,S_m)$. Let $x_C \in V(T)$ be such that $N_G(V(C)) \subseteq W_{x_C}$. Hence there is a k-dismantable tree decomposition $(T_C,(W_x^C \mid x \in V(T_C)))$ of $(C,S_m\cap V(C))$ of width at most $\mathrm{td}_k(G,S_m)-1$. We suppose that the trees T_C for C connected component of $G-\bigcup_{i\in[m-1]} S_i$ and T are pairwise disjoint. Let T' be the tree obtained from the union of T with all the T_C, by adding an edge between x_C and an arbitrary vertex in T_C, for each connected component C of $G-\bigcup_{i\in[m-1]} S_i$. For every $x\in V(T)$, let $W'_x = W_x$, and for every $x \in V(T_C)$, let $W'_x = W_x^C \cup W_{x_C}$, for each connected component C of $G-\bigcup_{i\in[m-1]} S_i$. See Figure 2.3. Then $(T',(W'_x \mid x \in V(T')))$ is a k-dismantable tree decomposition of (G,S) of width less than $\mathrm{td}_k(G,S_m)+\sum_{i\in[m-1]} \mathrm{td}_k(G,S_i)$. This proves the lemma. $\square$

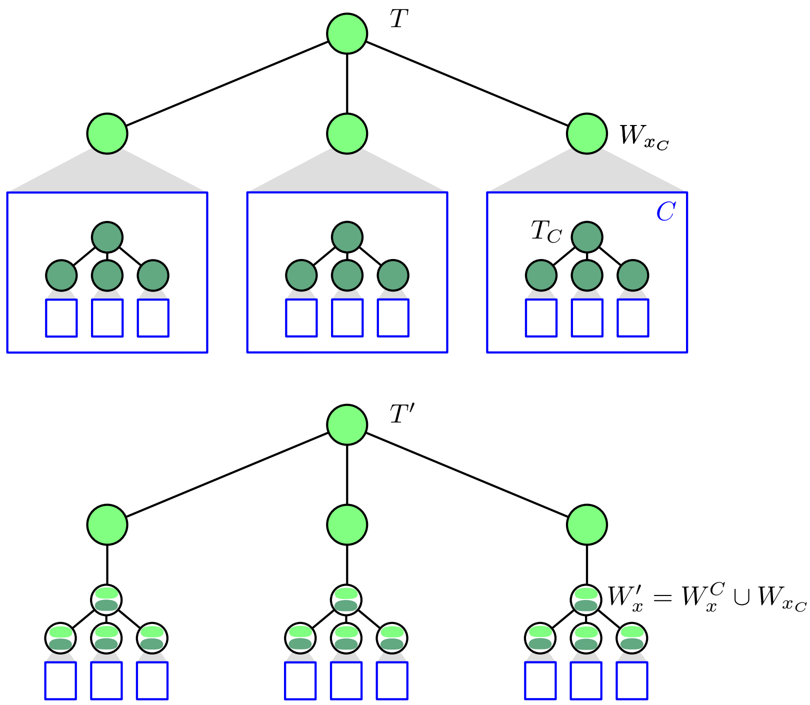


Figure 2.3: Illustration for the proof of Lemma 2.6.

2.3 $\{T\Box P_\ell \mid \ell \geqslant 1\}$ has unbounded k-treedepth

In this section, we prove that graphs of the form $T\Box P_\ell$ for some tree T on k vertices have unbounded k-treedepth when ℓ tends to $+\infty$. This proves the "only if" part of Theorem 1.20.

Proposition 2.7. *For every tree S, for every positive integer N, there is a positive integer ℓ such that*

$$\mathrm{td}_k(S\Box P_\ell) > N.$$

Proof. Fix a tree S on k vertices. Suppose for a contradiction that there exists a positive integer N such that $\mathrm{td}_k(S\Box P_\ell) \leqslant N$ for every positive integer ℓ. By taking N minimal, we can assume that, for some positive integer ℓ_0,

$$\mathrm{td}_k(S\Box P_\ell) = N$$

for every $\ell \geqslant \ell_0$.

Consider $G = S\Box P_{(2\ell_0+2)N}$. We denote by $R_1, \dots, R_k$ the copies of $P_{(2\ell_0+1)N}$ in G, that is the subgraphs $G[\{(x,i) \mid i \in [(2\ell_0+2)N]\}]$ for $x \in V(S)$; and by $C_1, \dots, C_{(2\ell_0+2)N}$ the copies of S in G, that is the subgraphs $G[\{(x,i) \mid x \in V(S)\}]$ for $i \in [(2\ell_0+2)N]$. By hypothesis $\mathrm{td}_k(G) = N$, and so by Observation 2.4, G admits a k-dismantable tree decomposition $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ of width $N-1$.

For every $x \in V(T)$, the bag W_x has size at most N, and so intersects at most N of the subgraphs C_i for $i \in [(2\ell_0+2)N]$. Therefore, by Lemma 1.17, there are indices $1 \leqslant i_1 < \dots < i_{2\ell_0+2} \leqslant (2\ell_0+2)N$ such that $C_{i_1}, \dots, C_{i_{2\ell_0+2}}$ have pairwise vertex disjoint projections on T, that is for every distinct $j, j' \in [2\ell_0+2]$, $T_j = T[\{x \in V(T) \mid W_x \cap C_{i_j} \neq \emptyset\}]$ and $T_{j'} = T[\{x \in V(T) \mid W_x \cap C_{i_{j'}} \neq \emptyset\}]$ are vertex-disjoint. For every $a \in [k]$ and $j \in [2\ell_0+1]$, let $R_{a,j}$ be the subpath of R_a with first vertex in C_{i_j} and last vertex in $C_{i_{j+1}}$. Moreover, let $u_{a,j}$ be the last vertex belonging to $\bigcup_{x\in V(T_j)} W_x \cap V(R_{a,j})$ along $R_{a,j}$. Note that for every $j \in [2\ell_0+1]$, $C'_j = \{u_{a,j} \mid a \in [k]\}$ is included in a bag of $\mathcal{D}$, namely W_{z_j} where z_j is the first node of the (unique) path from $V(T_j)$ to $V(T_{j+1})$ in T. Moreover, the vertices $u_{a,j}$ for $a \in [k]$ and $j \in [2\ell_0+1]$ are pairwise distinct.

Let F be the forest obtained from T by removing every edge $xy \in E(T)$ such that $|W_x \cap W_y| < k$. We claim that $\{z_j \mid j \in [2\ell_0+1]\}$ is in a single connected component of F. Indeed, otherwise there is a cut of size less than k between W_{z_j} and $W_{z_{j'}}$ in G, for some distinct $j, j' \in [2\ell_0+1]$. But this is a contradiction since $R_1, \dots, R_k$ induce k vertex-disjoint paths from $C'_j \subseteq W_{z_j}$ to $C'_{j'} \subseteq W_{z_{j'}}$. Let T' be this connected component.

Let $\hat{G}$ be the graph with vertex set $\bigcup_{z\in V(T')} W_z$ and edge set $\bigcup_{z\in V(T')} \binom{W_z}{2}$. For every $a \in [k]$, let $r_{a,1}, \dots, r_{a,m_a}$ be the vertices of $V(R_a) \cap V(\hat{G})$, in this order along R_a. Observe that $\hat{R}_a = (r_{a,1}, \dots, r_{a,m_a})$ is a path in $\hat{G}$. Moreover, for every $a \in [k]$, $\hat{R}_a$ intersects each of $C'_1, \dots, C'_{2\ell_0+1}$ in this order along $\hat{R}_a$. Since C'_j induces a clique in $\hat{G}$ for every $j \in [2\ell_0+1]$, we deduce that there is a model of a $K_k\Box P_{2\ell_0+1}$ in $\hat{G}$. Moreover, $\hat{G}$ has a k-dismantable tree decomposition $\mathcal{D}' = (T', (W_x \mid x \in V(T')))$ of width less than N and such that every adhesion has size at least k. It follows that there is a vertex $v \in V(\hat{G})$ such that $v \in W_x$ for every $x \in V(T')$ and $(T', (W_x \setminus \{v\} \mid x \in V(T')))$ is a k-dismantable tree decomposition. This implies that $\mathrm{td}_k(\hat{G} - v) < N$. But since $\hat{G}$ contains a model of $K_k\Box P_{2\ell_0+1}$, $\hat{G} - v$ contains a model of $K_k\Box P_{\ell_0}$. By Lemma 2.5, we deduce that

$$\mathrm{td}_k(S\Box P_{\ell_0}) \leqslant \mathrm{td}_k(K_k\Box P_{\ell_0}) \leqslant \mathrm{td}_k(\hat{G} - v) < N,$$

contradicting the definition of ℓ_0. This proves the lemma. □

2.4 Finding a k-ladder

In this section, we show how to extract a graph of the form $T \square P_\ell$ with T a tree on k vertices, from a graph having k disjoint paths $Q_1, \dots, Q_k$, and sufficiently many pairwise vertex-disjoint connected subgraphs intersecting all the Q_is.

Let k, ℓ be two positive integers. A *k-ladder of length $\ell - 1$* is a graph H with vertex set $[k] \times [\ell]$ such that

(i) for every $i \in [k]$, $R_i = H[\{(i,j) \mid j \in [\ell]\}]$ is a path with ordering $(i,1), \dots, (i,\ell)$, and

(ii) for every $j \in [\ell]$, $C_j = H[\{(i,j) \mid i \in [k]\}]$ is connected.

See Figure 2.4. We call $R_1, \dots, R_k$ the *rows* of H, and $C_1, \dots, C_\ell$ its *columns*.

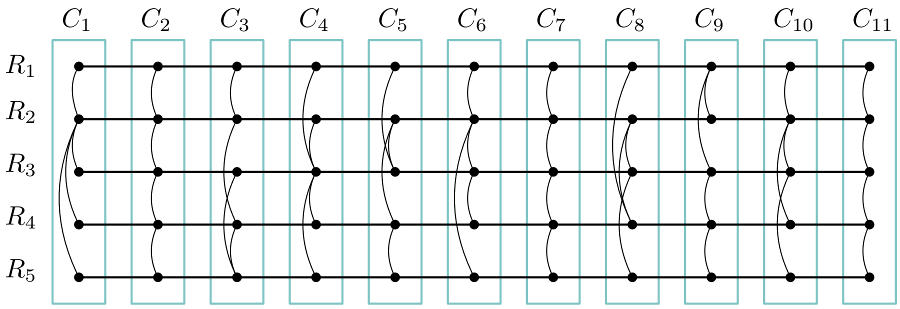


Figure 2.4: A 5-ladder of length 10.

Since every C_i is connected, we can assume without loss of generality that they are trees. Then, using Cayley's formula stating that there are k^{k-2} labelled trees on k vertices, we deduce the following observation.

Observation 2.8. *For every k, ℓ, if H is a k-ladder of length $k^{k-2}(\ell-1)+1$, then there is a tree T on k vertices such that $T \square P_\ell$ is a minor of H.*

Hence it suffices to find a long enough k-ladder as a minor to find a minor of the form $T \square P_\ell$ for some tree T on k vertices. Our main tool to do so is the following lemma, which is the main result of this section.

Lemma 2.9. *There is a function $f_{2.9} \colon \mathbb{N}^2 \to \mathbb{N}$ such that the following holds. Let k be a positive integer, let G be a graph, and let $Q_1, \dots, Q_k$ be k vertex-disjoint paths in G. If there exists $f_{2.9}(k,\ell)$ pairwise vertex-disjoint connected subgraphs in G each intersecting $V(Q_i)$ for every $i \in [k]$, then G has a k-ladder of length ℓ as a minor.*

Together with Observation 2.8, this lemma implies Lemma 2.1. To core of the proof of Lemma 2.9 is the following intermediary result.

Lemma 2.10. *There is a function $f_{2.10} \colon \mathbb{N}^2 \to \mathbb{N}$ such that the following holds. Let G be a graph, let t, ℓ be positive integers, let P be a path in G, and let $A_1, \dots, A_{f_{2.10}(t,\ell)}$ be pairwise vertex-disjoint connected subgraphs of G such that for every $i \in [f_{2.10}(t,\ell)]$, $1 \leqslant |V(A_i) \cap V(P)| \leqslant 2t-1$. Then there are subgraphs $B_1, \dots, B_\ell$ in $\{A_i\}_{i \in [f_{2.10}(t,\ell)]}$, a subpath of P of the form $P[\mathrm{init}(P), b]$, and for every $j \in [\ell]$ a subpath $P[a_j, b_j]$ of $P[\mathrm{init}(P), b]$, such that*

(a) $V(B_j) \cap V(P[\text{init}(P), b]) \neq \emptyset$ *for every* $j \in [\ell]$,

(b) $V(B_j) \cap V(P[\text{init}(P), b]) \subseteq V(P[a_j, b_j])$ *for every* $j \in [\ell]$, *and*

(c) $V(P[a_j, b_j]) \cap V(P[a_{j'}, b_{j'}]) = \emptyset$ *for every distinct* $j, j' \in [\ell]$.

Figure 2.5 gives an illustration for the outcome of this lemma.

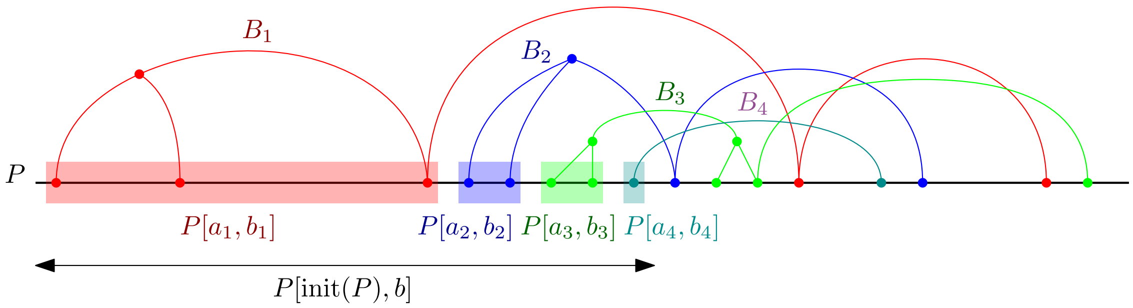


Figure 2.5: Illustration for Lemmas 2.10 and 2.12 with $\ell = 4$. There is a subpath $P[\text{init}(P), b]$ such that each B_j has a "private" interval $P[a_j, b_j]$ that contains $V(B_j) \cap V(P[\text{init}(P), b])$ and is disjoint from $V(P[a_{j'}, b_{j'}])$ for every $j' \neq j$.

Proof. Let $f_{2.10}(t, \cdot)$ be the function defined inductively by $f_{2.10}(t, 1) = 1$ and $f_{2.10}(t, \ell) = 1 + (2t-1)f_{2.10}(t, \ell - 1)$ for every $\ell \geqslant 2$. We proceed by induction on ℓ. If $\ell = 1$, then take for $B_1 = A_1$ and $P[\text{init}(P), b] = P[a_1, b_1] = P$.

Now suppose $\ell > 1$. Without loss of generality A_1 is such that the first vertex along P in $\bigcup_{i \in [f_{2.10}(t,\ell)]} V(A_i)$ belongs to A_1. Let $x_1, \ldots, x_r$ be the vertices in $V(A_1) \cap V(P)$, in this order along P. Note that $r \leqslant 2t - 1$ by hypothesis. Let x_{r+1} be the last vertex of P. For every $s \in [r]$, let I_s be the set of indices $j \in \{2, \ldots, f_{2.10}(k, \ell)\}$ such that the first vertex in $V(A_j)$ along P belongs to $P[x_s, x_{s+1}]$. By the pigeonhole principle, there is an integer $s \in [r]$ such that $|I_s| \geqslant \frac{f_{2.10}(t,\ell)-1}{2t-1} = f_{2.10}(t, \ell - 1)$.

Hence by the induction hypothesis applied to any family $\{A'_1, \ldots, A'_{f_{2.10}(k,\ell-1)}\} \subseteq \{A_j\}_{j \in I_s}$ and

$$P' = \begin{cases} P]x_s, x_{s+1}[& \text{if } s < r, \\ P]x_s, x_{s+1}] & \text{if } s = r, \end{cases}$$

there are subgraphs $B_2, \ldots, B_{\ell'}$ in $\{A'_i\}_{i \in [f_{2.10}(t,\ell')]}$ and a subpath $P]x_s, b']$ of P' such that for every $j \in \{2, \ldots, \ell'\}$, there is a subpath $P[a_j, b_j]$ of $P]x_s, b']$ such that

(a') $V(B_j) \cap V(P[\text{init}(P), b]) \neq \emptyset$, and

(b') $V(B_j) \cap V(P[\text{init}(P), b]) \subseteq V(P[a_j, b_j])$, and

(c') $V(P[a_{j'}, b_{j'}]) \cap V(P[a_{j''}, b_{j''}]) = \emptyset$ for every distinct $j', j'' \in \{2, \ldots, \ell\} \setminus \{j\}$.

Then for $B_1 = A_1$, $P[\text{init}(P), b] = P[\text{init}(P), b']$ and $P[a_1, b_1] = P[\text{init}(P), x_s]$, this proves the lemma. □

The following simple lemma will be useful.

Lemma 2.11. *For every positive integer k, and for every tree T of at least $(k-1)(k-2)+2$ vertices, at least one of the following holds*

(1) *T has at least k leaves, or*

(2) *T has a path on k vertices.*

Proof. If $k = 1$ then the result is clear. Now assume $k \geqslant 2$. Root T on an arbitrary vertex r, and suppose that T has at most $k-1$ leaves. Then for every leaf x, consider the path $T[x, r[$ be the path from x to r in T, minus r. Since $\bigcup_{x \text{ leaf}} V(T[x, r[)$ covers $V(T - r)$, by the pigeon hole principle, there is a leaf x such that $|V(T[x, r[)| \geqslant k - 1$ and so $|V(T[x, r])| \geqslant k$. □

We now show that the condition $|V(A_i) \cap V(P)| \leqslant 2t - 1$ in Lemma 2.10 can be removed, assuming that G has bounded treewidth.

Lemma 2.12. *There is a function $f_{2.12} \colon \mathbb{N}^2 \to \mathbb{N}$ such that the following holds. Let t, ℓ be positive integers, and let G be a graph with $\operatorname{tw}(G) < t$. If P is a path in G, and if $\{A_1, \ldots, A_{f_{2.12}(t,\ell)}\}$ is a family of pairwise vertex-disjoint connected subgraphs of G, all intersecting $V(P)$, then there exists ℓ subgraphs $B_1, \ldots, B_\ell$ among $\{A_1, \ldots, A_{f_{2.12}(t,\ell)}\}$ and a subpath of P of the form $P[\operatorname{init}(P), b]$ such that for every $j \in [\ell]$, there is a subpath $P[a_j, b_j]$ of $P[\operatorname{init}(P), b]$ such that*

(a) *$V(B_j) \cap V(P[\operatorname{init}(P), b]) \subseteq V(P[a_j, b_j])$, and*

(b) *$V(P[a_j, b_j]) \cap V(P[a_{j'}, b_{j'}]) = \emptyset$ for every $j' \in [\ell] \setminus \{j\}$.*

Proof. Let $f_{2.12}(t, \ell) = t \max\{2, f_{2.10}(t, \ell)\}^2$ where $f_{2.10}$ is as in Lemma 2.10. We can assume that if $u, v \in V(P) \cap V(A_i)$ are distinct and in this order along P, then $P[u, v]$ intersect A_j for some $j \neq i$, for every $i \in [f_{2.12}(t, \ell)]$. Otherwise we just contract $P[u, v]$ into a single vertex.

Consider now a tree decomposition $(T, (W_x \mid x \in V(T)))$ of G of width less than t. Since $A_1, \ldots, A_{f_{2.12}(t,\ell)}$ are pairwise vertex-disjoint, every set of vertices intersecting all of them must have size at least $f_{2.12}(t, \ell)$. Consider now for every $i \in [f_{2.12}(t, \ell)]$ the projection T_i of A_i on T. By Lemma 1.17, the family $(T_i)_{i \in [f_{2.12}(t,\ell)]}$ contains $N = \frac{f_{2.12}(t,\ell)}{t} = \max\{2, f_{2.10}(k, \ell)\}^2$ pairwise vertex-disjoint subtrees $T_{i_1}, \ldots, T_{i_N}$.

Consider now the tree T' obtained from T by contracting every vertex outside $\bigcup_{j \in [N]} V(T_{i_j})$ with one of its closest vertex in $\bigcup_{j \in [n]} V(T_{i_j})$, and by contracting T_{i_j} into a single node x_j for each $j \in [N]$. Hence $V(T') = \{x_j \mid j \in [N]\}$. By Lemma 2.11, T' has either $\sqrt{N} = f_{2.10}(k, \ell)$ leaves or a path with $\sqrt{N} = f_{2.10}(k, \ell)$ vertices. Let I be a subset of $[N]$ of size $\sqrt{N}$ such that x_j for $j \in I$ is a leaf in T', or such that a permutation of $(x_j)_{j \in I}$ is a path in T'. We claim that

$$\text{for every } j \in I, \text{ the number of pairs } u, v \in V(P) \text{ in this order along } P \text{ such that } u \in V(A_{i_j}),\ V(P]u, v[) \cap \Big(\bigcup_{j' \in I} V(A_{i_{j'}})\Big) = \emptyset, \text{ and } v \in \bigcup_{j' \in I \setminus \{j\}} V(A_{i_{j'}}), \text{ is at most } 2(t-1). \tag{2.2}$$

Indeed, if x_j is a leaf, then every time P leaves A_{i_j}, P crosses the adhesion between the root of T_{i_j} and its unique neighbor out of $V(T_{i_j})$, which have size at most $t - 1$. And if a permutation of $(x_{j'})_{j' \in I}$ is a path in T', then every time P leaves A_{i_j}, P must cross one of the at most two adhesions neighboring T_{i_j} and leading to another $T_{i_{j'}}$ for some $j' \in I \setminus \{j\}$. Since these adhesions have size at most $t - 1$, this proves (2.2).

Then contract repeatedly every edge of P that intersect at most one of the sets $V(A_{i_j})$ for $j \in I$. Let G' be the resulting minor of G, let P' be the path in G' induced by P, and for every $j \in I$, let A'_j be the connected subgraph of G' inherited from A_{i_j}. Then $|V(P') \cap A'_j| \leqslant 2t - 1$

for every $j \in I$. Hence we can apply Lemma 2.10 to $G', P', \{A_{i_j} \mid j \in I\}$. This is possible since $|I| = \sqrt{N} = g(t, \ell)$. We obtain subgraphs $B'_1, \ldots, B'_\ell$ of G' and a subpath $P'[\mathrm{init}(P'), b']$ of P' corresponding respectively to subgraphs $B_1, \ldots, B_\ell$ of G and a subpath $P[\mathrm{init}(P), b]$ of P as desired. This proves the lemma. □

To deduce Lemma 2.9, we will use the celebrated Erdős-Szekeres Theorem.

Theorem 2.13 (Erdős Szekeres [ES35]). *For every integers r, s, for every permutation σ of $[(r-1)(s-1)+1]$, either there is a set $I \subseteq [(r-1)(s-1)+1]$ of size r such that $\sigma|_I$ is increasing, or there is a set $D \subseteq [(r-1)(s-1)+1]$ of size s such that $\sigma|_D$ is decreasing.*

Proof of Lemma 2.9. Let $t = \max\{2, f_{1.10}(\max\{k, \ell\})\}$. If $\mathrm{tw}(G) \geqslant t$, then the $k \times \ell$ grid is a minor of G by Theorem 1.10, and so G contains a k-ladder of length ℓ as a minor. Now assume that $\mathrm{tw}(G) < t$. For every $i \in \{0, \ldots, k\}$, let

$$f_{i,k}(\ell) = \begin{cases} \ell^{2^{k-1}} & \text{if } i = 0, \\ f_{2.12}(t, f_{i-1,k}(\ell)) & \text{if } i \geqslant 1, \end{cases}$$

where $f_{2.12}$ is as in Lemma 2.12, and take $f_{2.9}(k, \ell) = f_{k,k}(\ell)$.

Let G be a graph, Let $Q_1, \ldots, Q_k$ in G be k disjoint paths in G, and let $A_1, \ldots, A_{f_{2.9}(k,\ell)}$ be pairwise disjoint connected subgraphs in G each intersecting $V(Q_i)$ for every $i \in [k]$, By applying Lemma 2.12 successively on the paths $Q_1, \ldots, Q_k$ and contracting the subpaths of the form $Q_i[a_j, b_j]$, we obtain a minor G' of G, connected subgraphs $B'_1, \ldots, B'_{\ell^{2^{k-1}}}$ of G', and paths $Q'_1, \ldots, Q'_k$ in G' such that

$$|V(B_j) \cap V(Q'_a)| = 1$$

for every $j \in [\ell^{2^{k-1}}]$ and $a \in [k]$. For every $j \in [\ell^{2^{k-1}}]$ and $a \in [k]$, let $u_{j,a}$ be the unique vertex in $V(B_j) \cap V(Q'_a)$. Finally, by applying Theorem 2.13 (Erdős-Szekeres Theorem) $k-1$ times, there are ℓ distinct indices $j_1, \ldots, j_\ell$ in $\{1, \ldots, \ell^{2^{k-1}}\}$ such that the orderings of $(u_{j_i,a})_{i \in [\ell]}$ along Q'_a for $a \in [k]$ are pairwise identical or reverse of each other. By possibly replacing Q'_{i_j} by its reverse for every $j \in [\ell]$, and by contracting some edges, we obtain a k-ladder of length ℓ as a minor of G' with rows inherited from $Q'_1, \ldots, Q'_k$ and columns inherited from $B_{j_1}, \ldots, B_{j_\ell}$. This proves the lemma. □

2.5 Nice pairs

In order to show that every graph G excluding all the k-ladders of length ℓ as minors have bounded k-treedepth, we need to find a decomposition of G into graphs $G_1, \ldots, G_m$, such that G is a $(< k)$-clique-sum of $G_1, \ldots, G_m$. Typically, G_i is the "torso" of a bag in a suitable tree decomposition. The main issue with this approach is that we then need to decompose each G_i, while G_i is not necessarily a minor of G (since we possibly added some cliques). The solution we develop in this section is to force G_i to be "nice" in G, which roughly means that the cliques added to $G[V(G_i)]$ to obtain G_i do not increase the connectivity between any two subsets of $V(G_i)$. In this section, we define this notion of "nice" sets, and we prove several properties on them. In the following section, we will find tree decompositions that will enable us to decompose our graph into nice sets in the final proof.

Let G be a graph. A *good pair* in G is a pair $(U, \mathcal{B})$ where U is a non-empty subset of $V(G)$, and $\mathcal{B}$ is a family of subsets of $V(G)$ such that (g1) $G = \bigcup_{B \in \mathcal{B} \cup \{U\}} G[B]$, and

(g2) the sets $B \setminus U$ for $B \in \mathcal{B}$ are pairwise disjoint.

Equivalently, $\mathcal{B} \cup \{U\}$ is the family of the bags of a tree decomposition indexed by a star whose center bag is U. Then, we say that $(U, \mathcal{B})$ is a *nice pair* in G if it is a good pair in G and

(g3) for every $B \in \mathcal{B}$, for every $i \geqslant 0$, for every $Z_1, Z_2 \subseteq U \cap B$ both of size i, there are i disjoint (Z_1, Z_2)-paths in $G[B] \setminus \binom{B \cap U}{2}$.

See Figure 2.6.

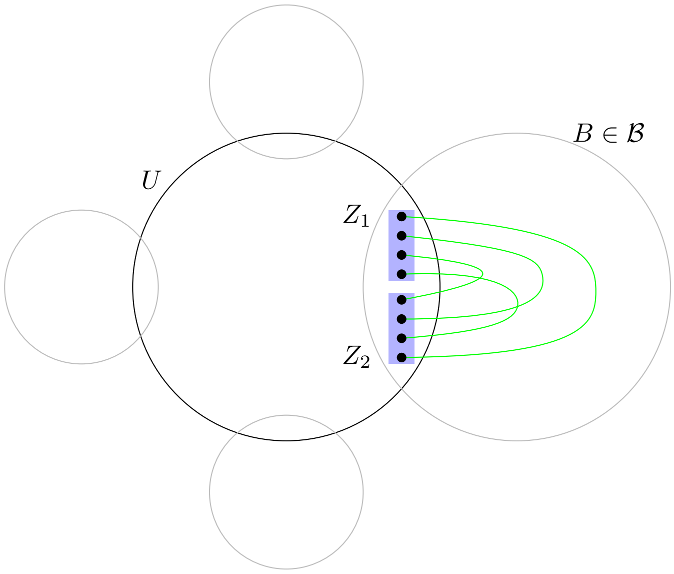


Figure 2.6: A good pair $(U, \mathcal{B})$ in G is nice if for every $B \in \mathcal{B}$, for every $i \geqslant 0$, for every $Z_1, Z_2 \subseteq U \cap B$ both of size i, there are i disjoint (Z_1, Z_2)-paths in $G[B] \setminus \binom{U \cap B}{2}$. This property implies that the maximum number of pairwise disjoint paths between two given subsets of U is the same in G and in $\text{torso}_G(U, \mathcal{B})$. See Lemma 2.17.

In practice, we will build good pairs from tree decompositions as follows. If $(T, (W_x \mid x \in V(T)))$ is a tree decomposition of a graph G, and if R is a subtree of T, then $(U, \mathcal{B})$ for $U = \bigcup_{x \in V(R)} W_x$ and $\mathcal{B} = \{\bigcup_{x \in V(S)} W_x \mid S \text{ connected component of } T - V(R)\}$ is a good pair in G. To ensure that $(U, \mathcal{B})$ is a nice pair, we will need further assumptions on the tree decomposition. Finding such decompositions will be the goal of Section 2.6, but for now, we prove some properties on nice pairs.

First we give some notation. Let G be a graph and let $(U, \mathcal{B})$ be a good pair in G. The *torso of U in G with respect to $\mathcal{B}$*, denoted by $\text{torso}_G(U, \mathcal{B})$, is the graph with vertex set U and edge set $E(G[U]) \cup \bigcup_{B \in \mathcal{B}} \binom{U \cap B}{2}$.

Lemma 2.14. *Let G be a graph and let $(U, \mathcal{B})$ be a nice pair in G. Then for every $X \subseteq U$, there exists a nice pair in $G - X$ of the form $(U \setminus X, \mathcal{B}')$ with* $\text{torso}_{G-X}(U \setminus X, \mathcal{B}') = \text{torso}_G(U, \mathcal{B}) - X$.

Proof. We claim that $(U \setminus X, \{B \setminus X \mid B \in \mathcal{B}\})$ is nice in $G - X$. First, (g1) and (g2) of the definition of good pair clearly hold, and so $(U \setminus X, \{B \setminus X \mid B \in \mathcal{B}\})$ is a good pair.

Consider now $B \in \mathcal{B}$, a positive integer i, and two sets $Z_1, Z_2 \subseteq (U \setminus X) \cap (B \setminus X)$ both of size i. Let $Z_0 = B \cap U \cap X$. Since $(U, \mathcal{B})$ is a nice pair, there are $i + |Z_0|$ disjoint $(Z_0 \cup Z_1, Z_0 \cup Z_2)$-paths in $G[B] \setminus \binom{B \cap U}{2}$. Note that $B \cap U = (U \setminus X) \cap (B \setminus X) \cup Z_0$. Hence there are i disjoint (Z_1, Z_2)-paths in $G[(U \setminus X) \cap (B \setminus X)] \setminus \binom{(U \setminus X) \cap (B \setminus X)}{2}$. This proves that $(U \setminus X, \{B \setminus X \mid B \in \mathcal{B}\})$ is a nice pair in $G - X$. Finally, by definition of torso, $\mathrm{torso}_G(U \setminus X, \{B \setminus X \mid B \in \mathcal{B}\}) = \mathrm{torso}_G(U, \mathcal{B}) - X$. □

Lemma 2.15. *Let G be a graph and let $(U, \mathcal{B})$ be a nice pair in G. For every $B \in \mathcal{B}$, $(V(G) \setminus (B \setminus U), \{B\})$ is a nice pair in G.*

Proof. Clearly, $(V(G) \setminus (B \setminus U), \{B\})$ is a good pair in G. Moreover, for every $i \geqslant 0$, for every $Z_1, Z_2 \subseteq U \cap B$ both of size i, there are i disjoint (Z_1, Z_2)-paths in $G[B] \setminus \binom{B \cap U}{2}$. Hence $(V(G) \setminus (B \setminus U), \{B\})$ is a nice pair in G. □

Lemma 2.16. *Let k' be a positive integer. Let G be a graph and let $(U, \mathcal{B})$ be a good pair in G. For every $Z_1, Z_2 \subseteq U$ with $|Z_1| = |Z_2| = k'$ such that there are k' disjoint (Z_1, Z_2)-paths $Q_1, \dots, Q_{k'}$ in G, there are k' disjoint (Z_1, Z_2)-paths $\hat{Q}_1, \dots, \hat{Q}_{k'}$ in $\mathrm{torso}_G(U, \mathcal{B})$. Moreover $\bigcup_{i \in [k']} V(Q_i) \cap U = \bigcup_{i \in [k']} V(\hat{Q}_i)$.*

Proof. Let $Z_1, Z_2 \subseteq U$ with $|Z_1| = |Z_2| = k'$ and let $Q'_1, \dots, Q'_{k'}$ be k' disjoint (Z_1, Z_2)-paths in G. Let $a \in [k']$ and let $q_{a,1}, \dots, q_{a,\ell_a}$ be the vertices in $V(Q_a) \cap U$ in this order along Q_a. Then, for every $i \in [\ell_a - 1]$, either $q_{a,i}q_{a,i+1}$ is an edge of G, or there exists $B \in \mathcal{B}$ such that $q_{a,i}q_{a,i+1} \in B$. In both cases $q_{a,i}a_{a,i+1}$ is an edge in $\mathrm{torso}_G(U, \mathcal{B})$. Hence $\hat{Q}_a = (q_{a,1}, \dots, q_{a,\ell_a})$ is a path in $\mathrm{torso}_G(U, \mathcal{B})$ with $V(\hat{Q}_a) = V(Q_a) \cap U$, and so $(\hat{Q}_1, \dots, \hat{Q}_{k'})$ is as claimed. □

The following lemma, which is the key property of nice pairs, informally says that if $(U, \mathcal{B})$ is nice, then the reciprocal of the previous lemma also holds.

Lemma 2.17. *Let k' be a positive integer. Let G be a graph and let $(U, \mathcal{B})$ be a nice pair in G. For every $Z_1, Z_2 \subseteq U$ with $|Z_1| = |Z_2| = k'$ such that there are k' disjoint (Z_1, Z_2)-paths $Q_1, \dots, Q_{k'}$ in $\mathrm{torso}_G(U, \mathcal{B})$, there are k' disjoint (Z_1, Z_2)-paths $Q'_1, \dots, Q'_{k'}$ in G. Moreover $\bigcup_{i \in [k']} V(Q'_i) \cap U \subseteq \bigcup_{i \in [k']} V(Q_i)$.*

Proof. We proceed by induction on $|\mathcal{B}|$. If $\mathcal{B} = \emptyset$, then $\mathrm{torso}_G(U, \mathcal{B}) = G$ and the result is clear. Now assume that $\mathcal{B} \neq \emptyset$ and consider some $B \in \mathcal{B}$. Note that $(U, \mathcal{B} \setminus \{B\})$ is a nice pair in $G' = \mathrm{torso}_G(V(G) \setminus B, \{B\})$. Hence by the induction hypothesis applied to $(U, \mathcal{B} \setminus \{B\})$ in G', there are k' disjoint (Z_1, Z_2)-paths $Q''_1, \dots, Q''_{k'}$ in G' with $\bigcup_{i \in [k']} V(Q''_i) \cap U \subseteq \bigcup_{i \in [k']} V(Q_i)$. Without loss of generality, assume that among $Q''_1, \dots, Q''_{k'}$, the paths intersecting $U \cap B$ in at least one vertex are $Q''_1, \dots, Q''_r$. By taking these paths induced in G', we can assume that $V(Q''_i) \cap N_G(V(C)) = \{z^i_1, z^i_2\}$ for some possibly equal $z^i_1, z^i_2 \in U \cap B$, for every $i \in [r]$. Moreover, we assume that if $z^i_1 \neq z^i_2$, then z^1_i is in the connected component of the end point of Q''_i belonging to Z_1 in $Q''_i - z^i_1 z^i_2$ (and so z^2_i is in the connected component of the end point of Q''_i belonging to Z_2 in $Q''_i - u$). Let $Z'_1 = \{z^i_1 \mid i \in [r]\}$, $Z'_2 = \{z^i_2 \mid i \in [r]\}$, and $Z'_0 = (U \cap B) \setminus (Z'_1 \cup Z'_2)$. By hypothesis, there are $r + |Z'_0|$ pairwise disjoint $(Z'_0 \cup Z'_1, Z'_0 \cup Z'_2)$-paths $Q'_1, \dots, Q'_r, (\{u\})_{u \in Z'_0}$ in $G[B] \setminus \binom{B \cap U}{2}$. Hence the union of $Q'_1, \dots, Q'_r$ with $Q''_1 - z^1_1 z^1_2, \dots, Q''_r - z^r_1 z^r_2$, together with $Q''_{r+1}, \dots, Q''_{k'}$, yields k' pairwise vertex-disjoint (Z_1, Z_2)-paths in G. Moreover $\bigcup_{i \in [k']} V(Q'_i) \cap U \subseteq \bigcup_{i \in [k']} V(Q_i)$. This proves the lemma. □

The following lemma informally says that a nice pair in the torso of a nice pair in G, is itself a nice pair in G.

Lemma 2.18. *Let G be a graph and let $(U, \mathcal{B})$ be a nice pair in G. If $(U', \mathcal{B}')$ is a nice pair in $\mathrm{torso}_G(U, \mathcal{B})$, then there exists $\mathcal{B}''$ such that $(U', \mathcal{B}'')$ is a nice pair in G, and $\mathrm{torso}_G(U', \mathcal{B}'') = \mathrm{torso}_{\mathrm{torso}_G(U,\mathcal{B})}(U', \mathcal{B}')$.*

Proof. We proceed by induction on $|\mathcal{B}|$. If $\mathcal{B}$ is empty, then $\mathrm{torso}_G(U, \mathcal{B}) = G$ and so $(U', \mathcal{B}')$ is a nice pair in G with $\mathrm{torso}_G(U', \mathcal{B}') = \mathrm{torso}_{\mathrm{torso}_G(U,\mathcal{B})}(U', \mathcal{B}')$. Now suppose that $\mathcal{B} \neq \emptyset$.

Consider a member B of $\mathcal{B}$. Let $G' = \mathrm{torso}_G(V(G) \setminus (B \setminus U), \{B\})$. Observe that $(U, \mathcal{B} \setminus \{B\})$ is a nice pair in G'. Hence, by the induction hypothesis, there is a nice pair in G' of the form $(U', \mathcal{B}''_0)$ with $\mathrm{torso}_{G'}(U', \mathcal{B}''_0) = \mathrm{torso}_{\mathrm{torso}_{G'}(U,\mathcal{B}\setminus\{B\})}(U', \mathcal{B}')$. By definition of G', the set $B \cap U$ induces a clique in G'. As $\{U'\} \cup \mathcal{B}''_0$ is the family of the bags of a tree decomposition of G', this implies that there exists $B''_0 \in \mathcal{B}''_0 \cup \{U'\}$ such that $B \cap U \subseteq B''_0$.

First suppose that $B''_0 \neq U$. Let

$$B''_1 = B \cup B''_0$$

and

$$\mathcal{B}'' = \mathcal{B}''_0 \setminus \{B''_0\} \cup \{B''_1\}.$$

By construction, $\mathrm{torso}_G(U', \mathcal{B}'') = \mathrm{torso}_{G'}(U', \mathcal{B}''_0) = \mathrm{torso}_{\mathrm{torso}_G(U,\mathcal{B})}(U', \mathcal{B}')$. It remains to show that $(U', \mathcal{B}'')$ is a nice pair in G. Observe that (g1) holds since $G' = \bigcup_{B' \in \mathcal{B}''_0 \cup \{U'\}} G'[B']$. Moreover, (g2) holds since B is disjoint from $B' \setminus U$ for all $B' \in \mathcal{B}''_0 \setminus \{B''_0\}$. We now prove (g3). Let $B'' \in \mathcal{B}''$, $i \geqslant 0$, and $Z_1, Z_2 \subseteq B'' \cap U'$ both of size i. By possibly replacing Z_a by $Z_a \cup (B'' \cap U' \setminus (Z_1 \cup Z_2))$ for each $a \in \{1, 2\}$ and increasing i accordingly, we now assume that $Z_1 \cup Z_2 = B'' \cap U'$. If $B'' \neq B''_1$, then there are i disjoint $(Z_1 \cap B''_0, Z_2 \cap B''_0)$-paths in $G'[B''] \setminus \binom{B'' \cap U}{2} = G[B''] \setminus \binom{B'' \cap U}{2}$ since $(U', \mathcal{B}''_0)$ is a nice pair in G'. If $B'' = B''_1$, then there are $|Z_1 \cap B''_0|$ disjoint (Z_1, Z_2)-paths $Q_1, \ldots, Q_i$ in $G'[B''_1] \setminus \binom{B''_1 \cap U}{2}$ since $(U', \mathcal{B}''_0)$ is a nice pair in G'. Without loss of generality, suppose that among $Q_1, \ldots, Q_i$, the paths $Q_1, \ldots, Q_j$, are the ones intersecting B. We can assume that these paths are induced in $G'[B''_1] \setminus \binom{B''_1 \cap U}{2}$. In particular, for every $a \in [j]$, Q_a intersects B in at most two vertices (because $Z_1 \cup Z_2 = B''_1 \cap U'$). For every $a \in [j]$, let z_1^a, z_2^a be respectively the first and last vertex of Q_a in $U \cap B$. Since $(U, \mathcal{B})$ is a nice pair in G, there are j disjoint $(\{z_1^a \mid a \in [j]\}, \{z_2^a \mid a \in [j]\})$-paths in $G[B] \setminus \binom{U \cap B}{2}$. Combining theses paths with $Q_1, \ldots, Q_i$, we obtain i disjoint (Z_1, Z_2)-paths in $G[B \cup B''_0] \setminus \binom{(B \cup B''_0) \cap U)}{2}$.

Now assume $B''_0 = U$. Let

$$\mathcal{B}'' = \mathcal{B}''_0 \cup \{B\}.$$

By construction of G', $\mathrm{torso}_G(U', \mathcal{B}'') = \mathrm{torso}_{G'}(U', \mathcal{B}''_0) = \mathrm{torso}_{\mathrm{torso}_G(U,\mathcal{B})}(U', \mathcal{B}')$. It remains to show that $(U', \mathcal{B}'')$ is a nice pair in G. Observe that (g1) holds since $G \subseteq G' \cup G[B]$ and $G' = \bigcup_{B' \in \mathcal{B}' \cup \{U'\}} G'[B']$. Moreover, (g2) holds since B is disjoint from all $B'' \setminus U$ for $B'' \in \mathcal{B}''_0$. We now prove (g3). Let $B'' \in \mathcal{B}''$, $i \geqslant 0$, and $Z_1, Z_2 \subseteq B'' \cap U'$ both of size i. If $B'' \neq B$, then there are i disjoint $(Z_1 \cap B'_0, Z_2 \cap B'_0)$-paths in $G'[B''] \setminus \binom{B'' \cap U}{2} = G[B''] \setminus \binom{B'' \cap U}{2}$ since $(U', \mathcal{B}''_0)$ is a nice pair in G'. If $B'' = B$, then there are $|Z_1 \cap B''_0|$ disjoint (Z_1, Z_2)-paths $Q_1, \ldots, Q_i$ in $G[B] \setminus \binom{B \cap U}{2}$ since $(U, \mathcal{B})$ is a nice pair in G. Hence (g3) holds. This concludes the proof of the lemma. □

2.6 Finding a suitable tree decomposition

In this section, we prove the existence of "well connected" tree decompositions, that will be used to decompose a graph into nice pairs in the final proof of Theorem 1.20. This is inspired by the seminal result of Thomas [Tho90], which asserts that every graph G admits a tree decomposition $(T,(W_x \mid x \in V(T)))$ of width $\mathrm{tw}(G)$ which is *lean*, that is such that for every $x_1, x_2 \in V(T)$, for every $Z_1 \subseteq W_{x_1}$ and $Z_2 \subseteq W_{x_2}$ of same size i, either there are i disjoint (Z_1, Z_2)-paths in G, or there exists $z_1z_2 \in E(T[x_1, x_2])$ such that $|W_{z_1} \cap W_{z_2}| < i$. Hence the connectivity between two bags is witnessed by the smallest adhesion between them. Here we prove analogous results for tree decompositions of (G, S), with some additional properties. The techniques we use are strongly inspired by the short proof of Thomas' theorem by Bellenbaum and Diestel [BD02].

First we start by defining a potential function on a tree decomposition. Let k be a positive integer, let G be a graph, and let $S \subseteq V(G)$. Let $\mathcal{D} = (T,(W_x \mid x \in V(T)))$ be a tree decomposition of (G, S). For every positive integer i, let $n_i(\mathcal{D}) = |\{x \in V(T) \mid |W_x| = i\}|$, that is the number of bags of size i in $\mathcal{D}$, and let

$$n_i^{(k)}(\mathcal{D}) = \begin{cases} |\{x \in V(T) \mid |W_x| = i\}| & \text{if } i > \frac{3}{2}(k-1), \\ |\{x \in V(T) \mid |W_x| = i \text{ and } \exists y \in N_T(x), |W_x \cap W_y| \geqslant k\}| & \text{if } i \leqslant \frac{3}{2}(k-1). \end{cases}$$

Then, let

$$n(\mathcal{D}) = (n_{|V(G)|}(\mathcal{D}), \dots, n_0(\mathcal{D}))$$

and

$$n^{(k)}(\mathcal{D}) = (n^{(k)}_{|V(G)|}(\mathcal{D}), \dots, n^{(k)}_0(\mathcal{D})).$$

We will consider these tuples in the lexicographic order. The following lemma shows how to improve $\mathcal{D}$ by reducing $n(\mathcal{D})$ or $n^{(k)}(\mathcal{D})$ if some connectivity property is not satisfied. Its proof follows step by step Bellenbaum and Diestel's argument [BD02].

Lemma 2.19. *Let a be a positive integer. Let G be a graph and let S be a nonempty subset of $V(G)$. Let $\mathcal{D} = (T,(W_x \mid x \in V(T)))$ be a tree decomposition of (G, S) of adhesion at most a. If there exists a positive integer i, two vertices $x_1, x_2 \in V(T)$, and two sets $Z_1 \subseteq W_{x_1}, Z_2 \subseteq W_{x_2}$ with $|Z_1| = |Z_2| = i$ such that there are no i disjoint (Z_1, Z_2)-paths in G, but $|W_z \cap W_{z'}| \geqslant i$ for every $zz' \in E(T[x_1, x_2])$, then there exist $S' \subseteq V(G)$ with $S \subseteq S'$ and a tree decomposition $\mathcal{D}' = (T',(W'_z \mid z \in V(T')))$ of (G, S') of adhesion at most $\max\{a, i-1\}$ such that $n(\mathcal{D}')$ is smaller than $n(\mathcal{D})$ for the lexicographic order. More precisely, there exist $z \in V(T[x_1, x_2])$ and $j_0 \in \{|W_z|, \dots, |V(G)|\}$ such that*

(a) *$n_j(\mathcal{D}') \leqslant n_j(\mathcal{D})$ for every $j \in \{j_0, \dots, |V(G)|\}$;*

(b) *$n_{j_0}(\mathcal{D}') < n_{j_0}(\mathcal{D})$; and*

(c) *if $x_1 = x_2$, then there are adjacent vertices z_1, z_2 in T' such that $Z_1 \subseteq W'_{z_1}$, $Z_2 \subseteq W'_{z_2}$, and $W'_{z_1} \cap W'_{z_2}$ is a minimum (Z_1, Z_2)-cut in G. In particular $|W'_{z_1} \cap W'_{z_2}| < i$.*

Moreover, for every positive integer k, if $|W_z| > \frac{3}{2}(k-1)$ or $\exists z' \in N_T(z), |W_z \cap W_{z'}| \geqslant k$, then there exists $j_1 \in \{|W_z|, \dots, |V(G)|\}$ such that

(d) $n_j^{(k)}(\mathcal{D}') \leqslant n_j^{(k)}(\mathcal{D})$ *for every* $j \in \{j_1, \dots, |V(G)|\}$*; and*

(e) $n_{j_1}^{(k)}(\mathcal{D}') < n_{j_1}^{(k)}(\mathcal{D})$.

Proof. Let $x_1, x_2 \in V(T)$ and let $Z_1 \subseteq W_{x_1}, Z_2 \subseteq W_{x_2}$ with $|Z_1| = |Z_2| = i$ such that there are no i disjoint (Z_1, Z_2)-paths in G but $|W_z \cap W_{z'}| \geqslant i$ for every $zz' \in E(T[x_1, x_2])$. Let (x_1', x_2') be such a pair of vertices in $V(T[x_1, x_2])$ with $\mathrm{dist}_T(x_1', x_2')$ minimum. For simplicity, assume that $x_1' = x_1$ and $x_2' = x_2$. For every $u \in V(G)$, let $z_u \in V(T)$ be such that $u \in W_{z_u}$ if $u \in S$, and otherwise such that W_{z_u} contains the neighborhood of the connected component of u in $G - S$. In both cases, pick such a z_u with $\mathrm{dist}_T(z_u, V(T[x_1, x_2]))$ minimum. Since there are no i disjoint (Z_1, Z_2)-paths in G, by Menger's Theorem, there exists a set $X \subseteq V(G)$ of size at most $i-1$ such that every (Z_1, Z_2)-path in G intersects X. Take such a set X of minimum size, and if equality, take the one minimizing $\sum_{u \in X} d_u$, where

$$d_u = \begin{cases} \mathrm{dist}_T\left(z_u, V(T[x_1, x_2])\right) & \text{if } u \in S, \\ \mathrm{dist}_T\left(z_u, V(T[x_1, x_2])\right) + 1 & \text{otherwise,} \end{cases}$$

for every $u \in X$.

Let C_1 be the union of all the connected components of $G - X$ that intersect Z_1, and let $C_2 = G - (X \cup V(C_1))$. Let $G_1 = G[V(C_1) \cup X]$ and $G_2 = G[V(C_2) \cup X]$. By Menger's Theorem, there is a family $(P_u \mid u \in X)$ of pairwise disjoint (Z_1, Z_2)-paths in G such that $u \in V(P_u)$ for every $u \in X$. For every $u \in X$, P_u is the union of two paths $P_u^1 \subseteq G - V(C_1)$ and $P_u^2 \subseteq G - V(C_2)$ meeting in exactly one vertex, namely u. For each $a \in \{1, 2\}$, let T^a be a copy of T with vertex set $\{z^a \mid z \in V(T)\}$ and edge set $\{z_1^a z_2^a \mid z_1 z_2 \in E(T)\}$, and let $\mathcal{D}_a = (T^a, (W_{z^a}^a \mid z^a \in V(T^a)))$ be the tree decomposition of $(G_a, (S \cap V(G_a)) \cup X)$ defined by

$$W_{z^a}^a = (W_z \cap V(G_a)) \cup \{u \in X \mid z \in V(T[z_u, x_{3-a}])\}$$

for every $z \in V(T)$.

Finally, let $S' = S \cup X$, $T' = T^1 \cup T^2 \cup \{x_1^2 x_2^1\}$, and let $W'_{z^a} = W^a_{z^a}$ for every $z \in V(T)$ and for every $a \in \{1, 2\}$. We claim that $\mathcal{D}' = \left(T', (W_z' \mid z \in V(T'))\right)$ satisfies the outcome of the lemma. First, observe that $\mathcal{D}'$ is a tree decomposition of (G, S'). Then, we claim that following property holds.

$$\text{For all } z \in V(T) \text{ and } a \in \{1, 2\},\ |W_{z^a}^a| \leqslant |W_z|. \tag{2.3}$$

To see this, observe that for every $u \in W_{z^a}^a \setminus W_z$, we have $u \in X$ and $W_z \cap V(P_u) \neq \emptyset$. By mapping u to a vertex in $W_z \cap V(P_u)$, we obtain an injection from $W_{z^a}^a \setminus W_z$ to $W_z \setminus W_{z^a}^a$. This shows that $|W_{z^a}^a \setminus W_z| \leqslant |W_z \setminus W_{z^a}^a|$ and so $|W_{z^a}^a| = |W_{z^a}^a \setminus W_z| + |W_{z^a}^a \cap W_z| \leqslant |W_z \setminus W_{z^a}^a| + |W_{z^a}^a \cap W_z| = |W_z|$.

The same argument also proves the following.

$$\text{For all } z_1 z_2 \in E(T) \text{ and } a \in \{1, 2\},\ |W_{z_1^a}^a \cap W_{z_2^a}^a| \leqslant |W_{z_1} \cap W_{z_2}|. \tag{2.4}$$

In particular, $\mathcal{D}'$ has adhesion at most $\max\{a, i-1\}$. We now investigate the equality case in (2.3).

$$\text{For all } z \in V(T) \text{ and } a \in \{1, 2\}, \text{ if } |W_{z^a}^a| = |W_z|, \text{ then } W_{z^{3-a}}^{3-a} \subseteq X. \tag{2.5}$$

Suppose for a contradiction that $|W_{z^a}^a| = |W_z|$ and $W_{z^{3-a}}^{3-a} \not\subseteq X$ for some $z \in V(T)$ and $a \in \{1,2\}$. Then W_z intersects $V(C_{3-a})$, and every in $W_z \cap V(C_{3-a})$ corresponds to a vertex $v \in X$ such that $z \in V(T[z_v, x_{3-a}])$. Let $Y = W_{z^a}^a \setminus W_z$ be the set of all these vertices. Then let

$$X' = (X \setminus Y) \cup (W_z \cap V(C_{3-a})).$$

Note that $|X'| = |X| < i$. We claim that X' intersects every $(Z_a \cup W_z, Z_{3-a})$-path in G. Let Q be a $(Z_a \cup W_z, Z_{3-a})$-path in G. Suppose for a contradiction that $V(Q) \cap X' = \emptyset$. Since $V(Q) \cap (X \cup (W_z \cap V(C_{3-a}))) \neq \emptyset$, we have $V(Q) \cap X \neq \emptyset$, and so $V(Q) \cap Y \neq \emptyset$. Let u_0 be the last vertex of $Y \cap V(Q)$ along Q. Then the subpath $Q]u_0, \mathrm{term}(Q)]$ of Q is included in C_{3-a}, and $Q[u_0, \mathrm{term}(Q)]$ is a $(W_{z_{u_0}}, W_{x_{3-a}})$-path. Since $z \in V(T[z_{u_0}, x_{3-a}])$, $Q[u_0, \mathrm{term}(Q)]$ intersects W_z. As $u_0 \notin W_z$, Q intersects W_z in $V(C_{3-a})$, and so $V(Q) \cap X' \neq \emptyset$. This proves that X' intersects every $(Z_a \cup W_z, Z_{3-a})$-path in G.

If $z \in V(T]x_a, x_{3-a}])$, then take any $Z \subseteq W_z$ of size i. Then X' intersects every (Z, Z_{3-a})-path in G, and so there are no i disjoint (Z, Z_{3-a})-paths in G. Therefore, the pair (z, x_{3-a}) contradicts the minimality of $\mathrm{dist}_T(x_1, x_2)$. Now suppose $z \notin V(T]x_a, x_{3-a}])$. We claim that $d_{u'} < d_u$ for every $u' \in X' \setminus X$ and for every $u \in X \setminus X' = Y$. Indeed, $z \in V(T[z_u, x_{3-a}])$ by definition of $W_{z^a}^a$, and as $z \notin V(T]x_a, x_{3-a}])$, z lies on the path from z_u to x_{3-a} in T. For both cases $u \in S$ and $u \notin S$, this implies that $d_u > \mathrm{dist}_T(z, V(T[x_1, x_2]))$. On the other hand, since $u' \in W_z$ and $z_{u'}$ is such that $u' \in W_{z_{u'}}$ and $\mathrm{dist}_T(z_{u'}, V(T[x_1, x_2]))$ is minimum, we have $d_{u'} \leqslant \mathrm{dist}(z, V(T[x_1, x_2]))$. It follows that $d_u > \mathrm{dist}_T(z, V(T[x_1, x_2])) \geqslant d_{u'}$ as claimed. Since $X' \setminus X$ and $X \setminus X'$ are nonempty, this contradicts the minimality of $\sum_{u \in X} d_u$. This proves (2.5).

We now prove the following property.

$$\text{There exists } z \in V(T[x_1, x_2]) \text{ such that } |W_{z^1}^1|, |W_{z^2}^2| < |W_z|. \tag{2.6}$$

By (2.5), it is enough to find $z \in V(T[x_1, x_2])$ such that W_z intersects both $V(C_1)$ and $V(C_2)$. Suppose for a contradiction that for every $z \in V(T[x_1, x_2])$, $W_z \setminus X \subseteq V(C_1)$ or $W_z \setminus X \subseteq V(C_2)$. Then there is an edge zz' in $T[x_1, x_2]$ such that $W_z \setminus X \subseteq V(C_1)$ and $W_{z'} \setminus X \subseteq V(C_2)$. It follows that $W_z \cap W_{z'} \subseteq X$ and so $|W_z \cap W_{z'}| < i$, a contradiction. This proves (2.6).

To conclude, fix z as in (2.6). Let $j_0 \in \{|W_z|, \ldots, |V(G)|\}$ be maximum such that there exists $z' \in V(T)$ satisfying $|W_{z'}| = j_0$ and $|W_{z'^1}^1|, |W_{z'^2}^2| < |W_{z'}|$. Then, since $|W_z| \geqslant |X| + 1$, (2.3) and (2.5) imply that $n_j(\mathcal{D}') \leqslant n_j(\mathcal{D})$ for every $j \in \{j_0, \ldots, |V(G)|\}$, and $n_{j_0}(\mathcal{D}') \leqslant n_{j_0}(\mathcal{D})$. This proves (a) and (b). Finally, if $x_1 = x_2$, then (c) follows from the definition of $\mathcal{D}'$ for $z_1 = x_1^2$ and $z_2 = x_2^1$. This proves the first part of the lemma.

We now prove the "moreover" part. Suppose that $|W_z| > \frac{3}{2}(k-1)$ or $\exists z' \in N_T(z), |W_z \cap W_{z'}| \geqslant k$. Then let $j_1 \in \{|W_z|, \ldots, |V(G)|\}$ be maximum such that there exists $z'' \in V(T)$ satisfying $|W_{z''}| = j_1$, $|W_{z''^1}^1|, |W_{z''^2}^2| < |W_{z''}|$, and $|W_{z''}| > \frac{3}{2}(k-1)$ or $\exists z' \in N_T(z''), |W_{z''} \cap W_{z'}| \geqslant k$. Just as before, (2.3), (2.4) and (2.5) imply that $n_j^{(k)}(\mathcal{D}') \leqslant n_j^{(k)}(\mathcal{D})$ for every $j \in \{j_1, \ldots, |V(G)|\}$, and $n_{j_1}^{(k)}(\mathcal{D}') < n_{j_1}^{(k)}(\mathcal{D})$. This proves (d) and (e). □

As a first application of this lemma, we show the following version of a result of Cygan, Komosa, Lokshtanov, Pilipczuk, Pilipczuk, Saurabh, and Wahlström [CKL+20].

Theorem 2.20. *Let k be a positive integer and let G be a graph. There is a tree decomposition $(T, (W_x \mid x \in V(T)))$ of G of adhesion at most $k-1$ such that*

(a) *for every integer i with $1 \leqslant i \leqslant k$, for every $x \in V(T)$, for every $Z_1, Z_2 \subseteq W_x$ with $|Z_1| = |Z_2| = i$, there are i pairwise disjoint (Z_1, Z_2)-paths in G; and*

(b) *if G is connected, then for every edge $x_1x_2 \in E(T)$, $\bigcup_{z \in V(T_{x_2|x_1})} W_z$ induces a connected subgraph of G.*

Proof. Let $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ be a tree decomposition of G of adhesion less than k. Note that such a tree decomposition always exists since the tree decomposition of G with a single bag $V(G)$ has adhesion less than k. We take such a tree decomposition $\mathcal{D}$ such that $n(\mathcal{D})$ is minimum for the lexicographic order.

First we show (a). Suppose for a contradiction that there exists i with $1 \leqslant i \leqslant k$, $x \in V(T)$, and $Z_1, Z_2 \subseteq W_x$ with $|Z_1| = |Z_2| = i$, such that there are no i pairwise disjoint (Z_1, Z_2)-paths in G. Then by Lemma 2.19 applied for $S = V(G)$, there exists $\mathcal{D}'$ a tree decomposition of G of adhesion at most $\max\{k-1, i-1\} = k-1$ such that $n(\mathcal{D}')$ is smaller than $n(\mathcal{D})$ for the lexicographic order. This contradicts the minimality of $n(\mathcal{D})$ and so (a) holds.

Now suppose that (b) does not hold. Then G is connected and there is an edge $x_1x_2 \in E(T)$ such that $G' = G\left[\bigcup_{z \in V(T_{x_2|x_1})} W_z\right]$ is not connected. Let C_1 be a connected component of G' and let $C_2 = G' - V(C_1)$. For each $a \in \{1, 2\}$, let T^a be a copy of $T_{x_2|x_1}$ with vertex set $\{z^a \mid z \in V(T_{x_2|x_1})\}$ and edge set $\{z_1^a z_2^a \mid z_1z_2 \in E(T_{x_2|x_1})\}$. Then, for every $z \in V(T_{x_2|x_1})$, let $W'_{z^a} = W_z \cap V(C_a)$, and for every $z \in V(T_{x_1|x_2})$, let $W'_z = W_z$. Finally, for $T' = T_{x_1|x_2} \cup T^1 \cup T^2 \cup \{x_1x_2^1, x_1x_2^2\}$, $\mathcal{D}' = (T', (W'_z \mid z \in V(T')))$ is a tree decomposition of G of adhesion at most $k-1$. We now prove that $n(\mathcal{D}')$ is smaller than $n(\mathcal{D})$, which contradicts the minimality of $n(\mathcal{D})$ and so proves the theorem. To do so, consider $z_0 \in V(T_{x_2|x_1})$ be such that W_{z_0} intersects both $V(C_1)$ and $V(C_2)$, and $|W_{z_0}|$ is maximum under this property. This is well-defined because W_{x_2} intersects both $V(C_1)$ and $V(C_2)$ since G is connected. Now, for every $z \in V(T_{x_2|x_1})$, if $|W_z| > |W_{z_0}|$, then by maximality of $|W_{z_0}|$, one of W'_{z^1} and W'_{z^2} is empty, and the other one is W_z. Moreover, if $|W_z| = |W_{z_0}|$, then either one of W'_{z^1} and W'_{z^2} is empty, and the other one is W_z; or W_z intersects both $V(C_1)$ and $V(C_2)$, and it follows that $|W'_{z^1}|, |W'_{z^2}| < |W_z| = |W_{z_0}|$. This second case occurs at least once for $z = z_0$. Altogether, this implies that $n_j(\mathcal{D}') \leqslant n_j(\mathcal{D})$ for every $j \in \{|W_{z_0}|+1, \ldots, |V(G)|\}$, and $n_{|W_{z_0}|}(\mathcal{D}') < n_{|W_{z_0}|}(\mathcal{D})$. Therefore, $n(\mathcal{D}')$ is smaller than $n(\mathcal{D})$ for the lexicographic order. This contradicts the minimality of $n(\mathcal{D})$ and so (b) holds. $\square$

We can now show the main result of this section.

Lemma 2.21. *Let k, a, t be positive integers with $a \geqslant k$. Let G be a graph and let $S \subseteq V(G)$. If there exists a tree decomposition of (G, S) of width less than t and adhesion at most a, then there exists $S' \subseteq V(G)$ with $S \subseteq S'$ and $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ a tree decomposition of (G, S') such that*

(a) *$\mathcal{D}$ has adhesion at most a;*

(b) *$\mathcal{D}$ has width less than $\max\left\{t, \frac{3}{2}(k-1)\right\}$;*

(c) *for every $x_1, x_2 \in V(T)$, for every $Z_1 \subseteq W_{x_1}$ and $Z_2 \subseteq W_{x_2}$ both of size k, either*

  (i) *there are k disjoint (Z_1, Z_2)-paths in G,*

  (ii) *there exists $z_1, z_2 \in E(T[x_1, x_2])$ with $|W_{z_1} \cap W_{z_2}| < k$, or*

  (iii) *$x_1 = x_2$, $|W_{x_1}| \leqslant \frac{3}{2}(k-1)$, and $|W_{x_1} \cap W_y| < k$ for every $y \in N_T(x_1)$;*

(d) *for every* $x_1x_2 \in E(T)$ *with* $|W_{x_1} \cap W_{x_2}| < k$, *if there exists* $x_3 \in N_T(x_2)$ *such that* $|W_{x_2} \cap W_{x_3}| \geqslant k$, *then for every positive integer* i, *for every* $Z_1, Z_2 \subseteq W_{x_1} \cap W_{x_2}$ *both of size* i, *there are* i *disjoint* (Z_1, Z_2)*-paths in*

$$G\left[\bigcup_{z \in V(T_{x_2|x_1})} W_z \cup \bigcup_{C \in \mathcal{C}(x_2|x_1)} V(C)\right] \setminus \binom{W_{x_1} \cap W_{x_2}}{2}$$

where $\mathcal{C}(x_2 \mid x_1)$ *is the family of all the connected components* C *of* $G - S'$ *such that* $N_G(V(C)) \subseteq \bigcup_{z \in V(T_{x_2|x_1})} W_z$ *and* $N_G(V(C)) \not\subseteq W_{x_1} \cap W_{x_2}$.

See Figure 2.7 for an illustration of (d).

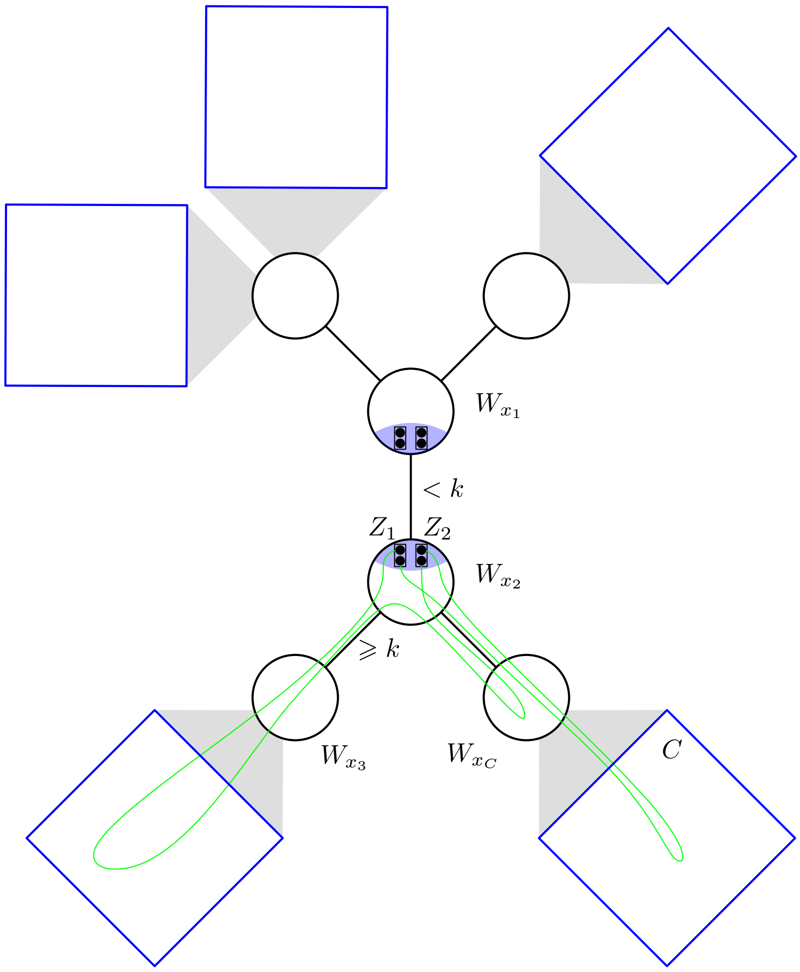


Figure 2.7: Illustration for Item (d) in Lemma 2.21.

Proof. Let $S' \subseteq V(G)$ containing S and $\mathcal{D}' = (T, (W_x \mid x \in V(T)))$ be a tree decomposition of (G, S') of adhesion at most a and width less than $\max\{t, \frac{3}{2}(k-1)\}$. Consider such a pair $(S', \mathcal{D}')$ such that $n^{(k)}(\mathcal{D}') = (n^{(k)}_{|V(G)|}(\mathcal{D}'), \ldots, n^{(k)}_0(\mathcal{D}'))$ is minimum for the lexicographic order. We claim that $\mathcal{D}'$ satisfies (a)-(d). By definition, (a) and (b) hold.

Now suppose that (c) does not hold: there exist $x_1, x_2 \in V(T)$, $Z_1 \subseteq W_{x_1}$ and $Z_2 \subseteq W_{x_2}$ both of size k, such that

(i) there are no k disjoint (Z_1, Z_2)-paths in G;

(ii) for every $z_1, z_2 \in E(T[x_1, x_2])$, $|W_{z_1} \cap W_{z_2}| \geqslant k$; and

(iii) $x_1 \neq x_2$, or $|W_{x_1}| > \frac{3}{2}(k-1)$, or there exists $y \in N_T(x_1)$ such that $|W_{x_1} \cap W_y| \geqslant k$.

Then by Lemma 2.19, there exists a set $S'' \subseteq V(G)$ containing S' (and so S) and a tree decomposition $\mathcal{D}''$ of (G, S'') of adhesion at most $\max\{a, k-1\} = a$ such that 2.19.(a)-2.19.(e) hold. If $x_1 \neq x_2$, then, since $z \in V(T[x_1, x_2])$, this implies that z has a neighbor z' with $|W_{z'} \cap W_{z'}| \geqslant k$. Hence, by 2.19.(d) and 2.19.(e), $n^{(k)}(\mathcal{D}'')$ is smaller than $n^{(k)}(\mathcal{D}')$ for the lexicographic order, a contradiction. Now suppose $x_1 = x_2$. If $|W_{x_1}| > \frac{3}{2}(k-1)$, then again by 2.19.(d) and 2.19.(e), $n^{(k)}(\mathcal{D}'')$ is smaller than $n^{(k)}(\mathcal{D}')$ for the lexicographic order, a contradiction. Now suppose $|W_{x_1}| \leqslant \frac{3}{2}(k-1)$. This implies that there exists $y \in N_T(x_1)$ such that $|W_{x_1} \cap W_y| \geqslant k$. Then again by 2.19.(d) and 2.19.(e), $n^{(k)}(\mathcal{D}'')$ is smaller than $n^{(k)}(\mathcal{D}')$ for the lexicographic order, a contradiction. This proves that (c) holds.

Now suppose that (d) does not hold. Hence there exist $x_1x_2 \in E(T)$ with $|W_{x_1} \cap W_{x_2}| < k$, $x_3 \in N_T(x_2)$ with $|W_{x_2} \cap W_{x_3}| \geqslant k$, a positive integer i, and two sets $Z_1, Z_2 \subseteq W_{x_1} \cap W_{x_2}$ both of size i, such that there are no i disjoint (Z_1, Z_2)-paths in $G_0 = G\left[\bigcup_{z \in V(T_{x_2|x_1})} W_z \cup \bigcup_{C \in \mathcal{C}(x_2|x_1)} V(C)\right] \setminus \binom{W_{x_1} \cap W_{x_2}}{2}$. By possibly replacing Z_a by $Z_a \cup ((W_{x_1} \cap W_{x_2}) \setminus (Z_1 \cup Z_2))$ for every $a \in \{1, 2\}$, and i by $i + |(W_{x_1} \cap W_{x_2}) \setminus (Z_1 \cup Z_2)|$, we assume that $Z_1 \cup Z_2 = W_{x_1} \cap W_{x_2}$.

We call Lemma 2.19 on G_0, $S_0 = V(G_0) \cap S'$, the tree decomposition $\mathcal{D}_0 = (T_{x_2|x_1}, (W_z \mid x \in V(T_{x_2|x_1})))$, x_2, x_2, Z_1, Z_2. We deduce that there exists $S_0' \subseteq V(G_0)$ with $S_0 \subseteq S_0'$ and a tree decomposition $\mathcal{D}_0' = (T_0, (W_z' \mid z \in V(T')))$ of (G_0, S_0') of adhesion at most $\max\{a, i-1\}$ such that

2.19.(c) there are adjacent vertices z_1, z_2 in T' such that $Z_1 \subseteq W_{z_1}'$, $Z_2 \subseteq W_{z_2}'$, and $W_{z_1}' \cap W_{z_2}'$ is a minimum (Z_1, Z_2)-cut in G. In particular $|W_{z_1}' \cap W_{z_2}'| < i$.

Moreover, there exists $j_1 \in \{|W_{x_2}|, \ldots, |V(G)|\}$ such that

2.19.(d) $n_j^{(k)}(\mathcal{D}_0') \leqslant n_j^{(k)}(\mathcal{D}_0)$ for every $j \in \{j_1, \ldots, |V(G)|\}$; and

2.19.(e) $n_{j_1}^{(k)}(\mathcal{D}_0') < n_{j_1}^{(k)}(\mathcal{D}_0)$.

Now let T_1 be obtained from T_0 by subdividing once the edge z_1, z_2. Let z_0 be the resulting new vertex. Then let $W_{z_0}' = Z_1 \cup Z_2 \cup (W_{z_1}' \cap W_{z_2}')$. Since $W_{z_1}' \cap W_{z_2}'$ is a minimum (Z_1, Z_2)-cut, we have

$$|(W_{z_1}' \cap W_{z_2}') \setminus (Z_1 \cap Z_2)| \leqslant \min\{|Z_1 \setminus Z_2|, |Z_2 \setminus Z_1|\} \leqslant \frac{k-1}{2}.$$

This implies that $|W_{z_0}'| \leqslant \frac{3}{2}(k-1)$. Moreover,

$$\begin{aligned}
|W_{z_0}' \cap W_{z_a}'| &= |(W_{z_1}' \cap W_{z_2}') \cup Z_a| \\
&\leqslant |(W_{z_1}' \cap W_{z_2}') \setminus (Z_1 \cap Z_2)| + |Z_a| \\
&\leqslant |Z_{3-a} \setminus Z_a| + |Z_a| \\
&= |Z_1 \cup Z_2| < k
\end{aligned}$$

for every $a \in \{1,2\}$. Then let $S'' = (S' \setminus V(G_0)) \cup S'_0$, and let $\mathcal{D}'' = (T'', (W''_z \mid z \in V(T'')))$ where $V(T'') = V(T_{x_1|x_2}) \cup V(T_1)$, $E(T'') = E(T_{x_1|x_2}) \cup E(T_1) \cup \{x_1z_0\}$, $W''_z = W_z$ for every $z \in V(T_{x_1|x_2})$, and $W''_z = W'_z$ for every $z \in V(T_1)$. See Figure 2.8. Then $\mathcal{D}''$ is a tree decomposition of (G, S'') of adhesion at most $\max\{a, i-1\}$ and width less than $\max\left\{t, \frac{3}{2}(k-1)\right\}$. Moreover, since every neighbor z of z_0 is such that $|W''_{z_0} \cap W''_z| < k$, the bag W''_{z_0} will not be counted in $n^{(k)}(\mathcal{D}'')$, and it follows that $n_j^{(k)}(\mathcal{D}'') \leqslant n_j^{(k)}(\mathcal{D}')$ for every $j \in \{j_1, \ldots, |V(G)|\}$ and $n_{j_1}^{(k)}(\mathcal{D}'') < n_{j_1}^{(k)}(\mathcal{D}')$. Therefore, $n^{(k)}(\mathcal{D}'')$ is smaller than $n^{(k)}(\mathcal{D}')$ for the lexicographic order, a contradiction. This proves that (d) holds. □

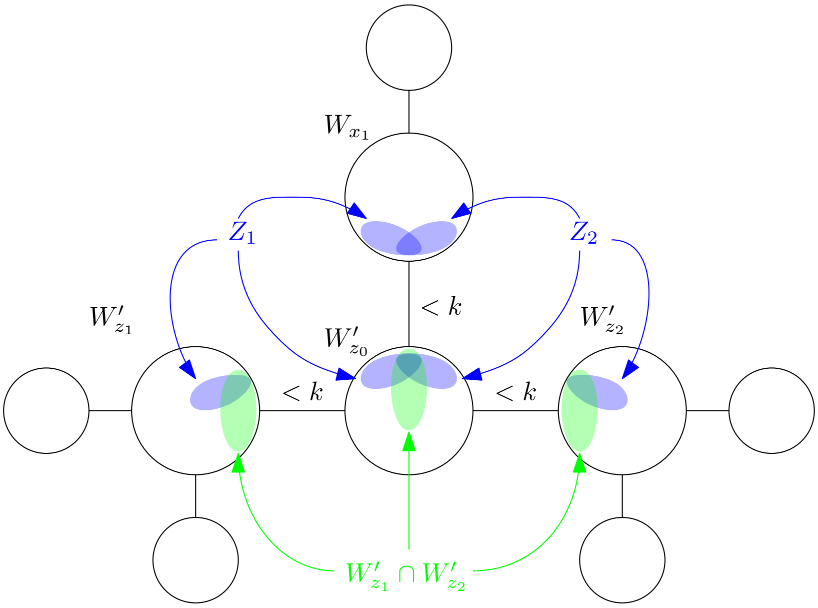


Figure 2.8: Construction of $\mathcal{D}''$ in the proof of Lemma 2.21.

2.7 Graphs excluding all k-ladders of length ℓ have bounded k-treedepth

In this section, we prove the “if” part of Theorem 1.20, that is the fact that graphs excluding all k-ladders of length ℓ as minors have bounded k-treedepth. To do so, we will prove the following statement by induction.

Theorem 2.22. *There is a function $f_{2.22} \colon \mathbb{N}^4 \to \mathbb{N}$ such that the following holds. Let k, ℓ, a, t be positive integers. For every graph G that does not contain any k-ladder of length ℓ as a minor, for every nice pair $(U, \mathcal{B})$ in G, if $S \subseteq U$ is such that there is a tree decomposition of $(\mathrm{torso}_G(U, \mathcal{B}), S)$ of adhesion at most a and width less than t, then there exists S' with $S \subseteq S' \subseteq U$ such that*

$$\mathrm{td}_k(\mathrm{torso}_G(U, \mathcal{B}), S') \leqslant f_{2.22}(k, \ell, t, a).$$

To deduce Theorem 1.20, observe that Proposition 2.7 implies that every minor-closed class of graphs of bounded k-treedepth excludes $T \square P_\ell$ for every tree T on k vertices, for some integer

ℓ. For the other direction, if $\mathcal{C}$ is a minor-closed class of graphs that does not contain $T \square P_\ell$ for every tree T on k vertices, then Observation 2.8 implies that no graph in $\mathcal{C}$ contains a k-ladder of length $k^{k-2}(\ell-1)+1$ as a minor, and Theorem 1.10 implies that every graph in G has treewidth less than $f_{1.10}(\max\{k,\ell\})$. Hence Theorem 2.22 applied for $U = S = V(G)$ and $\mathcal{B} = \emptyset$ implies that a graph in such a class has k-treedepth at most $f_{2.22}(k, k^{k-2}(\ell-1)+1, f_{1.10}(\max\{k,\ell\}), f_{1.10}(\max\{k,\ell\}))$.

Before showing Theorem 2.22, we give some notation and prove a key lemma. Let G be a graph, let $X \subseteq V(G)$ and let $u \in V(G)$. The *projection* of u on X is the set

$$\Pi_G(X,u) = \begin{cases} \{u\} & \text{if } u \in X \\ N_G(V(C)) & \text{if } u \notin X \text{ and } C \text{ is the connected component of } u \text{ in } G - X. \end{cases}$$

Moreover, for every $A \subseteq V(G)$, let $\Pi_G(X,A) = \bigcup_{u \in A} \Pi_G(X,u)$. Observe that for every connected subgraph H of $G - \Pi_G(X,A)$ intersecting X, $V(H)$ is disjoint from A.

The following lemma is inspired by Huynh, Joret, Micek, Seweryn, Wollan's proof for the case $k = 2$ [HJM+21].

Lemma 2.23. *There is a function $f_{2.23}\colon \mathbb{N}^3 \to \mathbb{N}$ such that the following holds. Let k, ℓ, c be positive integers. Let G be a connected graph that does not contain any k-ladder of length ℓ as a minor, let $(U, \mathcal{B})$ be a nice pair in G, and let $R \subseteq U$ with $|R| \leqslant c$. There is a set $Z \subseteq U$ with $R \subseteq Z$ such that*

(a) *for every connected component C of $\mathrm{torso}_G(U,\mathcal{B}) - Z$, R is connected in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$; and*

(b) $\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B}), Z) \leqslant f_{2.23}(k,\ell,c)$.

Proof. Let

$$f_{2.23}(k,\ell,c) = c + (c-1)\cdot \max\{2(k-1)+(2k-1)(k-1),\\ 2(k-1) + t\cdot (f_{2.9}(k,\ell)\cdot f_{1.10}(\max\{k,\ell\})+2k)\}.$$

Let G be a connected graph that does not contain any k-ladder of length ℓ, let $(U,\mathcal{B})$ be a nice pair in G, and let $R \subseteq U$ with $|R| \leqslant c$. Fix an ordering $r_1, \dots, r_{|R|}$ of R. Note that since G is connected, $\mathrm{torso}_G(U,\mathcal{B})$ is connected by Lemma 2.17. By Theorem 2.20 applied to $\mathrm{torso}_G(U,\mathcal{B})$, there is a tree decomposition $(T, (W_x \mid x \in V(T)))$ of $\mathrm{torso}_G(U,\mathcal{B})$ of adhesion at most $k-1$ such that

2.20.(a) for every integer i with $1 \leqslant i \leqslant k$, for every $x \in V(T)$, for every $Z_1, Z_2 \subseteq W_x$ with $|Z_1| = |Z_2| = i$, there are i pairwise disjoint (Z_1,Z_2)-paths in $\mathrm{torso}_G(U,\mathcal{B})$; and

2.20.(b) for every edge $x_1x_2 \in E(T)$, the set $\bigcup_{z \in V(T_{x_1|x_2})} W_z$ induces a connected subgraph of $\mathrm{torso}_G(U,\mathcal{B})$.

Fix some $i \in [|R|-1]$. We will build a set $Z_i \subseteq V(G)$ containing r_i and r_{i+1} such that

(a1) for every connected component C of $G - Z_i$, r_i and r_{i+1} are connected in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$; and

(a2) $\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B}), Z_i) \leqslant \max\{2(k-1)+(2k-1)(k-1), 2(k-1)+t\cdot(f_{2.9}(k,\ell)\cdot f_{1.10}(\max\{k,\ell\})+2k)\}$.

Let $x_1, x_2 \in V(T)$ be such that $r_i \in W_{x_1}$ and and $r_{i+1} \in W_{x_2}$. Consider some $y \in V(T[x_1,x_2])$. We will build a set $Z_{i,y} \subseteq W_y$ such that (b1) for every connected component C of $\mathrm{torso}_G(U,\mathcal{B}) - Z_{i,y}$ intersecting W_y, r_i and r_{i+1} are connected in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$;

(b2) $|Z_{i,y}| \leqslant \max\{2(k-1)+(2k-1)(k-1), 2(k-1)+t\cdot(f_{2.9}(k,\ell)\cdot f_{1.10}(\max\{k,\ell\})+2k)\}$; and

(b3) $Z_{i,y}$ contains the adhesions of y with its neighbors in $T[x_1,x_2]$, that is $W_y \cap W_{y'} \subseteq Z_{i,y}$ for every $y' \in N_{T[x_1,x_2]}(y)$.

Let A_1 be the adhesion of W_y with its predecessor in $T[x_1,x_2]$ if $y \neq x_1$, and let $A_1 = \{r_i\}$ if $y = x_1$. Similarly, let A_2 be the adhesion of W_y with its successor in $T[x_1,x_2]$ if $y \neq x_2$, and let $A_2 = \{r_{i+1}\}$ if $y = x_2$. Let Q be a path from A_1 to A_2 in $\mathrm{torso}_G(U,\mathcal{B})$. Such a path exists by 2.20.(a).

If $|V(Q) \cap W_y| < 2k$, then let

$$Z_{i,y} = A_1 \cup A_2 \cup \Pi_{\mathrm{torso}_G(U,\mathcal{B})}(W_y, V(Q)).$$

Note that $|Z_{i,y}| \leqslant 2(k-1) + (2k-1)(k-1)$, and for every connected component C of $\mathrm{torso}_G(U,\mathcal{B}) - Z_{i,y}$ intersecting W_y, $V(C)$ is disjoint from $V(Q)$. Hence (b1), (b2), and (b3) hold by construction.

Now suppose that $|V(Q) \cap W_y| \geqslant 2k$. Let B_1 be the vertex set of the shortest prefix of Q containing k vertices of W_y, and let B_2 be the vertex set of the shortest suffix of Q containing k vertices in W_y. By 2.20.(a), there are k pairwise vertex-disjoint $(B_1 \cap W_y, B_2 \cap W_y)$-paths in $\mathrm{torso}_G(U,\mathcal{B})$. By Lemma 2.17, there are k pairwise vertex-disjoint (B_1,B_2)-paths $Q_1,\dots,Q_k$ in G. Let $\mathcal{F}$ be the family of all the connected subgraphs H of G such that $V(H) \cap Q_j \neq \emptyset$ for every $j \in [k]$. By Lemma 2.9, there are no $f_{2.9}(k,\ell)$ pairwise vertex-disjoint members of $\mathcal{F}$. Hence by Lemma 1.17 and Theorem 1.10, there is a set $Z^0_{i,y} \subseteq V(G)$ of size at most $f_{2.9}(k,\ell)\cdot f_{1.10}(\max\{k,\ell\})$ that intersects every member of $\mathcal{F}$. Then let

$$Z_{i,y} = A_1 \cup A_2 \cup \Pi_G(W_y, Z^0_{i,y} \cup B_1 \cup B_2).$$

See Figure 2.9.

Note that (b3) holds by construction, and $Z_{i,y} \subseteq W_y$. Moreover, for every vertex u of G, $|\Pi_G(W_y,u)| \leqslant t$ since $t \geqslant 1$ and for every connected component C of $G - U$, $N_G(V(C))$ induces a clique in $\mathrm{torso}_G(U,\mathcal{B})$, and so has size at most t. It follows that

$$|Z_{i,y}| \leqslant 2(k-1) + t\cdot(f_{2.9}(k,\ell)\cdot f_{1.10}(\max\{k,\ell\})+2k),$$

which proves (b2). Consider now a connected component C of $\mathrm{torso}_G(U,\mathcal{B}) - Z_{i,y}$ intersecting W_y. Let C' be the connected component of $G - Z_{i,y}$ containing $V(C)$. Then $V(C')$ is disjoint from $Z^0_{i,y} \cup B_1 \cup B_2$. In particular, $C' \notin \mathcal{F}$ and so $V(C')$ is disjoint from Q'_j for some $j \in [k]$. Fix such a $j \in [k]$. By Lemma 2.16, there exists a path $\hat{Q}_j$ from $\mathrm{init}(Q_j)$ to $\mathrm{term}(Q_j)$ in $\mathrm{torso}_G(U,\mathcal{B})$ with vertex set $V(Q_j) \cap U$. Hence $\hat{Q}_j$ is a path from B_1 to B_2 in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$. Extending it using $G[B_1]$ and $Q[B_2]$, we obtain a path from A_1 to A_2 in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$. If $y = x_1$, then recall that $A_1 = \{r_i\}$. If $y \neq x_1$, let y' the predecessor of y along $T[x_1,x_2]$. Then $\bigcup_{z \in V(T_{y'|y})} W_z$ induces a connected subgraph of $\mathrm{torso}_G(U,\mathcal{B})$, and this subgraph is disjoint from $V(C)$ and contains r_i. In both cases, there is path from r_i to every vertex in A_1 in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$. Similarly, there is a path between r_{i+1} and every vertex in A_2 in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$. We deduce that there is a path from r_i to r_{i+1} in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$. This proves (b1).

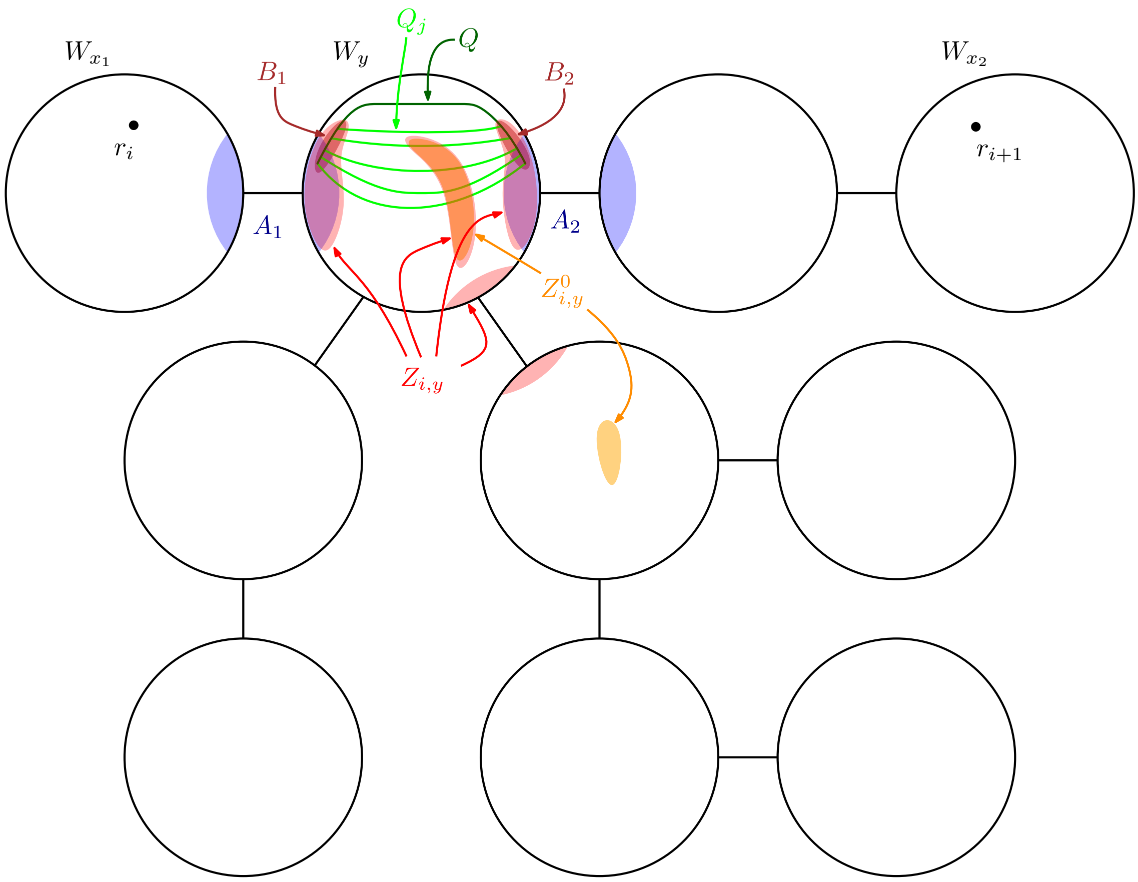


Figure 2.9: Illustration for the proof of Lemma 2.23. For sake of clarity, we assume here that $U = V(G)$, and Q, B_1, and B_2 are drawn inside W_y, but in general this is not the case, and so $Z_{i,y}$ contains only the projection of $B_1 \cup B_2$ on W_y.

Now let

$$Z_i = \bigcup_{y \in V(T[x_1,x_2])} Z_{i,y}.$$

By (b3), $(T[x_1,x_2], (Z_{i,y} \mid y \in V(T[x_1,x_2])))$ is a tree decomposition (and even a path decomposition) of $(\text{torso}_G(U,\mathcal{B}) - R, Z_i)$ of adhesion at most $k-1$ and of width less than $\max\{2(k-1)+(2k-1)(k-1), 2(k-1)+(k-1)\cdot(f_{2.9}(k,\ell)\cdot f_{1.10}(\max\{k,\ell\})+2k-1)\}$. Hence

$$\begin{aligned}\text{td}_k(\text{torso}_G(U,\mathcal{B}) - R, Z_i) \leqslant \max\{&2(k-1)+(2k-1)(k-1),\\ &2(k-1)+t\cdot(f_{2.9}(k,\ell)\cdot f_{1.10}(\max\{k,\ell\})+2k)\},\end{aligned}$$

and so (a1) holds. Moreover, for every connected component C of $\text{torso}_G(U,\mathcal{B}) - R - Z_i$, if $V(C)$ intersects W_y for some $y \in V(T[x_1,x_2])$, then by (b1), there is a path from r_i to r_{i+1} in $\text{torso}_G(U,\mathcal{B}) - V(C)$. Otherwise, there is an edge $yy' \in E(T)$ with $y \in V(T[x_1,x_2])$ and $y' \notin V(T[x_1,x_2])$ such that $V(C) \subseteq \bigcup_{z\in V(T_{y'|y})} W_z \setminus W_y$. Then, since $\bigcup_{z\in V(T_{y|y'})} W_z$ induces a connected subgraph of $\text{torso}_G(U,\mathcal{B})$ which contains r_i and r_{i+1} and is disjoint from $V(C)$, there is a path from r_i to r_{i+1} in $\text{torso}_G(U,\mathcal{B}) - V(C)$. Therefore, (a1) holds.

Finally, let

$$Z = R \cup \bigcup_{i\in[|R|-1]} Z_i.$$

Then for every connected component C of $\mathrm{torso}_G(U,\mathcal{B}) - Z$ for every $i \in [|R|-1]$, C is disjoint from Z_i and so by (a1), there is a path from r_i to r_{i+1} in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$. Therefore, R is connected in $\mathrm{torso}_G(U,\mathcal{B}) - V(C)$, which proves (a). Moreover, by Lemma 2.6,

$$\begin{aligned}\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B}), Z) &\leqslant |R| + (|R|-1)\cdot \max\{2(k-1)+(2k-1)(k-1),\\ &\qquad\qquad t\cdot f_{2.9}(k,\ell)\cdot f_{1.10}(\max\{k,\ell\})+2k\}\\ &\leqslant f_{2.23}(k,\ell,c),\end{aligned}$$

and so (b) holds. This proves the lemma. □

We can now give the proof of Theorem 2.22.

Proof of Theorem 2.22. Let k, ℓ be fixed positive integers. We proceed by induction on a. If $a \leqslant k-1$, then $(\mathrm{torso}_G(U,\mathcal{B}), S)$ has a tree decomposition of width less than t and adhesion at most $k-1$, and so $\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B}), S) \leqslant t \leqslant f_{2.22}(k,\ell,t,a)$ for $f_{2.22}(k,\ell,t,a) = t$. Now suppose $a \geqslant k$, and that the result holds for $a-1$.

First, we prove the following claim.

Claim 2.22.1. *There is a function $f_{2.22.1}\colon \mathbb{N}^4 \to \mathbb{N}$ such that the following holds. Let ℓ', c, c' be positive integers. Let G be a graph that does not contain any k-ladder of length ℓ as a minor. Let $(U,\mathcal{B})$ be a nice pair in G. Let $S \subseteq U$ be such that there is a tree decomposition of $(\mathrm{torso}_G(U,\mathcal{B}), S)$ of adhesion at most a and width less than t. Let $(V_0,\dots,V_{\ell'})$ be a path partition of $\mathrm{torso}_G(U,\mathcal{B})$ such that*

(a) $N_{\mathrm{torso}_G(U,\mathcal{B})}(V_i) \cap V_{i-1}$ *is connected in* $\mathrm{torso}_G(U,\mathcal{B})[(N_{\mathrm{torso}_G(U,\mathcal{B})}(V_i)\cap V_{i-1}) \cup V_i]$, *for every* $i \in [\ell'-1]$;

(b) $|N_{\mathrm{torso}_G(U,\mathcal{B})}(V_{\ell'})| \leqslant c$*; and*

(c) *for every $W \subseteq V_{\ell'}$, either*

  (i) *there are k disjoint (V_0, W)-paths in* $\mathrm{torso}_G(U,\mathcal{B})$*, or*

  (ii) $\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B})[W], S\cap W) \leqslant c'$.

Then there exists $S' \subseteq V_{\ell'}$ with $S \cap V_{\ell'} \subseteq S'$ such that

$$\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B})[V_{\ell'}], S') \leqslant f_{2.22.1}(t,\ell',c,c').$$

Before proving it, we motivate the statement of this claim. Its proof will consists in a downward induction on ℓ'. The path partition $(V_0,\dots,V_{\ell'})$ will ensure that the induction will eventually stop: if this path partition is long enough, and if we have k disjoint paths from V_0 to $V_{\ell'}$, then we will find a long k-ladder by Lemma 2.9, since each part is morally connected by (a). The problem with this approach is that there are graphs, even with large k-treedepth, that have no path partition with more than 3 parts. This is for example the case when there is a universal vertex. But in this specific case, removing the universal vertex will decrease the treewidth by one. In general, we will use the fact mentioned in Section 2.1 that for every graph G, for every $u \in V(G)$, there exists

$S \subseteq V(G) \setminus \{u\}$ containing $N_G(u)$ with $\mathrm{tw}(G-u,S) < \mathrm{tw}(G)$. Hence, by induction, we can decompose a superset of $N_G(u)$ in $G-u$, which will enable us to construct a path partition. This is why we are working in the setting of decomposition "focused" on a subset S of vertices.

Proof of the claim. Let $L = f_{2.9}(k,\ell)$, where $f_{2.9}$ is the function from Lemma 2.9. If $\ell' \geqslant 2L$, then let

$$f_{2.22.1}(t,\ell',c,c') = c'.$$

Otherwise, if $\ell' < 2L$, let

$$\begin{aligned} f_{2.22.1}(t,\ell',c,c') = f_{2.23}(k,\ell,c) & \\ &+ c\cdot(t+1) \\ &+ c\cdot f_{2.22}(k,\ell,t-1,a-1)+ \\ &+ f_{2.22.1}(t,\ell'+1, \\ &\qquad f_{2.23}(k,\ell,c)+c\cdot(t+1)+f_{2.22}(k,\ell,t-1,a-1),c'). \end{aligned}$$

We proceed by induction on $(2L-\ell',|V_{\ell'}|)$, in the lexicographic order. Let $\mathcal{D}_0 = (T_0,(W^0_x \mid x\in V(T_0)))$ be a tree decomposition of $(\mathrm{torso}_G(U,\mathcal{B}),S)$ of adhesion at most a and width less than t. If $\ell' \geqslant 2L$, then, by (c), either

(i) there are k disjoint (V_0,V_{2L})-paths in $\mathrm{torso}_G(U,\mathcal{B})$, or

(ii) $\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B})[V_{\ell'}],S\cap V_{\ell'}) \leqslant c'$.

In the second case we are done. Now assume that there are k disjoint $(V_0,V_{\ell'})$-paths in $\mathrm{torso}_G(U,\mathcal{B})$, and so k disjoint (V_0,V_{2L})-paths in $\mathrm{torso}_G(U,\mathcal{B})$. By Lemma 2.17, there are k disjoint (V_0,V_{2L})-paths $Q_1,\dots,Q_k$ in G. Moreover, since $(V_0,\dots,V_{2L})$ is a path partition of $\mathrm{torso}_G(U,\mathcal{B})$, every (V_0,V_{2L})-path in G intersects $N_{\mathrm{torso}_G(U,\mathcal{B})}(V_i)\cap V_{i-1}$ for every $i\in[2L]$. For every $i\in[2L]$, $N_{\mathrm{torso}_G(U,\mathcal{B})}(V_i)\cap V_{i-1}$ is connected in $\mathrm{torso}_G(U,\mathcal{B})[(N_{\mathrm{torso}_G(U,\mathcal{B})}(V_i)\cap V_{i-1})\cup V_i]$ by (a). Let C_i be the vertex set of the connected component of $\mathrm{torso}_G(U,\mathcal{B})[(N_G(V_i)\cap V_{i-1})\cup V_i]$ containing $N_G(V_i)\cap V_{i-1}$. Let $\mathcal{C}_i$ be the family of all the members B of $\mathcal{B}$ with $B\cap V(C_i)\neq\emptyset$. Not that $\bigcup\mathcal{C}_i$ and $\bigcup\mathcal{C}_j$ are disjoint for every $i,j\in[2L]$ with $|i-j|\geqslant 2$, because $B\cap U$ induces a clique in $\mathrm{torso}_G(U,\mathcal{B})$, for every $B\in\mathcal{B}$. Moreover, $N_{\mathrm{torso}_G(U,\mathcal{B})}(V_i)\cap V_{i-1}$ is connected in $G[C_i\cup\bigcup\mathcal{C}_i]$. Let C'_i be the connected component of $G[C_i\cup\bigcup\mathcal{C}_i]$ containing $N_G(V_i)\cap V_{i-1}$. Then $C'_1,C'_3,\dots,C'_{2L-1}$ is a family of L pairwise disjoint connected subgraphs of G, each of them intersecting $V(Q_j)$ for every $j\in[k]$. By Lemma 2.17, Lemma 2.9, and the definition of L, this implies that G contains a k-ladder of length ℓ as a minor, a contradiction.

Now suppose $\ell' < 2L$. If $\mathrm{torso}_G(U,\mathcal{B})[V_{\ell'}]$ is not connected, then let $\mathcal{C}$ be the family of all the connected components of $G[V_{\ell'}]$. For every $C\in\mathcal{C}$, we call induction on the path partition $(V_0,\dots,V_{\ell'-2},V_{\ell'-1}\cup(V_{\ell'}\setminus V(C)),V(C))$. We deduce that there exists $S'_C\subseteq V(C)$ containing $S\cap V(C)$ such that $\mathrm{td}_k(C,S'_C)\leqslant f_{2.22.1}(t,\ell',c,c')$. Then for $S'=\bigcup_{C\in\mathcal{C}}S'_C$, we have $\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B})[V_{\ell'}],S')\leqslant f_{2.22.1}(t,\ell',c,c')$. Now assume that $\mathrm{torso}_G(U,\mathcal{B})[V_{\ell'}]$ is connected.

A general picture of the following proof is provided in Figure 2.10. Let $X = V_1\cup\dots\cup V_{\ell'-2}\cup(V_{\ell'-1}\setminus N_{\mathrm{torso}_G(U,\mathcal{B})}(V_{\ell'}))$, and let $U'=U\setminus X$. Note that $U' = N_{\mathrm{torso}_G(U,\mathcal{B})}(V_{\ell'})\cup V_{\ell'}$. By Lemma 2.14, there exists a nice pair $(U',\mathcal{B}')$ in $G-X$ with

$$\mathrm{torso}_{G-X}(U',\mathcal{B}') = \mathrm{torso}_G(U,\mathcal{B}) - X = \mathrm{torso}_G(U,\mathcal{B})[N_{\mathrm{torso}_G(U,\mathcal{B})}(V_{\ell'})\cup V_{\ell'}].$$

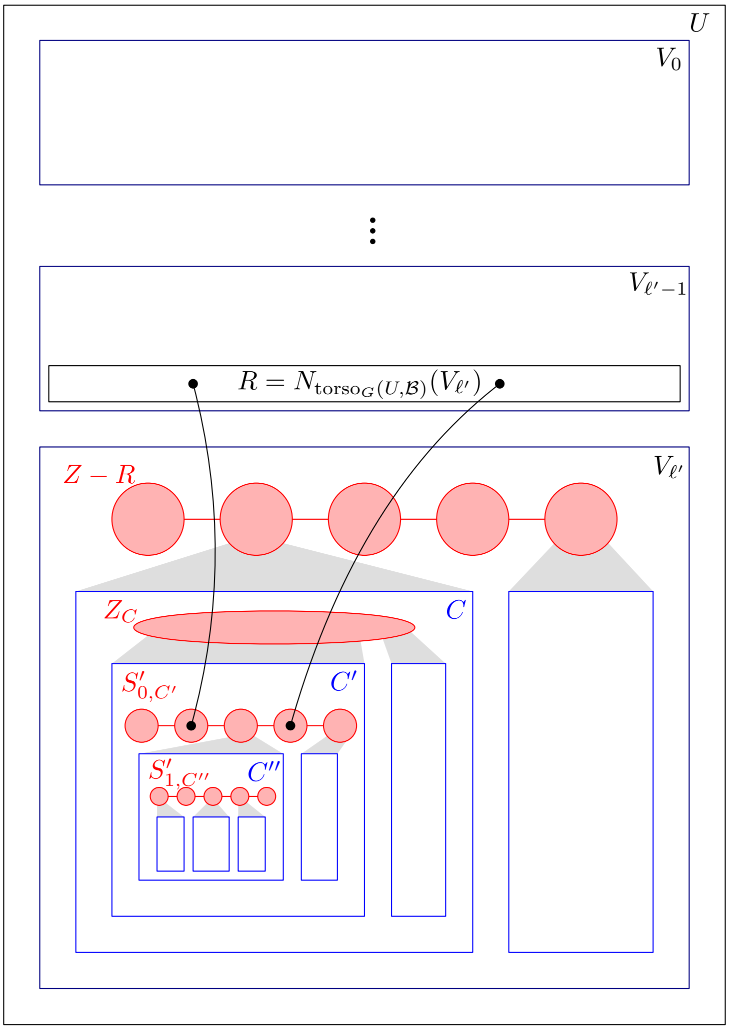


Figure 2.10: Illustration for the proof of Claim 2.22.1. First we remove the set Z given by Lemma 2.23 to ensure that R is connected in $\mathrm{torso}_{G'}(U', \mathcal{B}') - V(C)$ for every connected component C of $\mathrm{torso}_{G'}(U', \mathcal{B}') - Z$. Fix a connected component C of $\mathrm{torso}_{G'}(U', \mathcal{B}')$. Then, we remove Z_C to ensure that connected components of $C - Z_C$ are either disjoint from S (and so we are done), or have their neighborhood included in S. Fix a connected component C' of $C - Z_C$ which intersects S. We want to call induction on ℓ' by considering the partition $(V_0, \dots, V_{\ell'-1}, V_{\ell'} \setminus V(C'), V(C'))$. However, this might be not a path partition, since $V(C')$ can have neighbors in R. Hence we need to decompose $N_{\mathrm{torso}_{G'}(U',\mathcal{B}')}(u) \cap V(C')$ for every $u \in R$. This is done by calling induction on a, the adhesion of the given tree decomposition of $(\mathrm{torso}_G(U, \mathcal{B}), S)$. Now that the neighborhood of R is covered by a set $S'_{0,C'}$, we can indeed call induction on ℓ', which yields $S'_{1,C''}$ for each component C'' of $C' - S'_{0,C'}$ intersecting S. Then, combining the k-dismantable tree decompositions obtained at each step, we deduce the claim.

Moreover, U' induces a connected subgraph of $\mathrm{torso}_G(U,\mathcal{B})$ disjoint from X, and so by Lemma 2.17, U' is connected in $G-X$. Let G' be the connected component of $G-X$ containing U'. By possibly replacing every member B of $\mathcal{B}'$ by $B \cap V(G')$, we can assume that $\bigcup \mathcal{B}' \subseteq V(G')$. Then, $(U',\mathcal{B}')$ is a nice pair in G'. By Lemma 2.23 applied to $R = N_{\mathrm{torso}_G(U,\mathcal{B})}(V_{\ell'})$, there is a set $Z \subseteq U'$ containing R such that

2.23.(a) for every connected component C of $\mathrm{torso}_{G'}(U',\mathcal{B}') - Z$, the set R is connected in $\mathrm{torso}_{G'}(U',\mathcal{B}') - V(C)$; and

2.23.(b) $\mathrm{td}_k(\mathrm{torso}_{G'}(U',\mathcal{B}'),Z) \leqslant f_{2.23}(k,\ell,c)$.

Let $\mathcal{D}_Z = (T_Z,(W^Z_x \mid x \in V(T_Z)))$ be a k-dismantable tree decomposition of $(\mathrm{torso}_{G'}(U',\mathcal{B}'),Z)$ of width less than $f_{2.23}(k,\ell,c)$. Recall that $\mathrm{torso}_{G'}(U',\mathcal{B}') - Z = \mathrm{torso}_G(U,\mathcal{B}) - (X \cup Z)$. Fix a connected component C of $\mathrm{torso}_G(U,\mathcal{B}) - (X \cup Z)$. For every $u \in (N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \cap R) \setminus S$, let C_u be the connected component of u in $\mathrm{torso}_G(U,\mathcal{B}) - S$. Since $\mathcal{D}_0$ is a tree decomposition of $(\mathrm{torso}_G(U,\mathcal{B}),S)$, there exists $x_u \in V(T_0)$ such that $N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C_u)) \subseteq W^0_{x_u}$. Now let

$$Z_C = \bigcup_{u \in \left(N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \cap R\right) \setminus S} \left(W^0_{x_u} \cap V(C)\right).$$

Note that $Z_C \subseteq V(C) \cap S$ and $|Z_C| \leqslant c \cdot (t+1)$. Now, for every connected component C' of $C - Z_C$, either $V(C') \cap S = \emptyset$, or $N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C')) \cap R \subseteq S$. Let $\mathcal{C}_C$ be the family of all the connected components of $C - Z_C$ intersecting S. Fix $C' \in \mathcal{C}_C$, and consider some vertex $u \in N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C')) \cap R$. Recall that $u \in S$. Let $T_{0,u}$ be the subtree of T_0 induced by $\{x \in V(T_0) \mid u \in W^0_x\}$, and let $S_u = \bigcup_{x \in V(T_{0,u})} W^0_x \cap V(C')$. Then $(T_{0,u},(W^0_x \cap V(C') \mid x \in V(T_{0,u})))$ is a tree decomposition of (C',S_u) of width less than $t-1$ and adhesion at most $a-1$ since $u \in W^0_x \setminus V(C')$ for every $x \in V(T_{0,u})$. Moreover, by Lemma 2.14, since $V(C') = U \setminus (U \setminus V(C'))$, there is a nice pair in $G - (U \setminus V(C'))$ of the form $(V(C'),\mathcal{B}_{C'})$ such that $C' = \mathrm{torso}_G(U,\mathcal{B}) - (U \setminus V(C')) = \mathrm{torso}_G(V(C'),\mathcal{B}_{C'})$. Hence, by the induction hypothesis applied to $G - (U \setminus V(C')), V(C'), \mathcal{B}_{C'}, S_u$, there exists $S'_u \subseteq V(C')$ with $S_u \subseteq S'_u$ such that

$$\mathrm{td}_k(C',S'_u) \leqslant \mathrm{td}_k\left(\mathrm{torso}_{G-(U\setminus V(C'))}(V(C'),\mathcal{B}_{C'}),S'_u\right) \leqslant f_{2.22}(k,\ell,t-1,a-1).$$

Let

$$S'_{0,C'} = \bigcup_{u \in N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C')) \cap R} S'_u.$$

Then, by Lemma 2.6,

$$\mathrm{td}_k(C',S'_{0,C'}) \leqslant c \cdot f_{2.22}(k,\ell,t-1,a-1).$$

Consider now a connected component C'' of $C' - S'_{0,C'}$ with $V(C'') \cap S \neq \emptyset$. Since C'' is disjoint from $\bigcup_{u \in N_{\mathrm{torso}_{G'}(U',\mathcal{B}')}(V(C')) \cap R} S_u$, this implies that there is no edge between R and $V(C'')$ in $\mathrm{torso}_G(U,\mathcal{B})$. Moreover,

$$|N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C''))| \leqslant c \cdot f_{2.23}(k,\ell,c) + c \cdot (t+1) + f_{2.22}(k,\ell,t-1,a-1).$$

Then $(V_0, \dots, V_{\ell'-1}, V_{\ell'} \setminus V(C''), V(C''))$ is a path partition of $\mathrm{torso}_G(U, \mathcal{B})$. By the properties of Z, $R = N_{\mathrm{torso}_G(U,\mathcal{B})}(V_{\ell'})$ is connected in the graph

$$\begin{aligned}\mathrm{torso}_{G'}(U', \mathcal{B}') - V(C'') &= \mathrm{torso}_G(U, \mathcal{B}) - X - V(C'') \\ &= \mathrm{torso}_G(U, \mathcal{B})\big[N_{\mathrm{torso}_G(U,\mathcal{B})}(V_{\ell'}) \cup (V_{\ell'} \setminus V(C''))\big].\end{aligned}$$

Hence $(V_0, \dots, V_{\ell'-1}, V_{\ell'} \setminus V(C''), V(C''))$ satisfies the hypothesis of the claim for the parameters $(k, \ell, t, a, \ell' + 1, f_{2.23}(k, \ell, c) + c \cdot (t + 1) + f_{2.22}(k, \ell, t - 1, a - 1), c')$. Hence, by induction on ℓ', there exists $S'_{1,C''} \subseteq V(C'')$ with $S \cap V(C'') \subseteq S'_{1,C''}$ such that

$$\mathrm{td}_k(C'', S'_{1,C''}) \leqslant f_{2.22.1}(t, \ell' + 1, f_{2.23}(k, \ell, c) + c \cdot (t + 1) + f_{2.22}(k, \ell, t - 1, a - 1), c').$$

Now, let

$$S'_{C'} = S'_{0,C'} \cup \bigcup_{\substack{C'' \text{ connected components of } C' - S'_{0,C} \\ V(C'') \cap S \neq \emptyset}} S'_{1,C''}.$$

Then, by Lemma 2.6,

$$\begin{aligned}\mathrm{td}_k(C', S'_{C'}) &\leqslant \mathrm{td}_k(C', S'_{0,C'}) + \max_{\substack{C'' \text{ connected components of } C' - S'_{0,C} \\ V(C'') \cap S \neq \emptyset}} \mathrm{td}_k(C'', S'_{1,C''}) \\ &\leqslant c \cdot f_{2.22}(k, \ell, t - 1, a - 1) + \\ &\quad + f_{2.22.1}(t, \ell' + 1, \\ &\qquad f_{2.23}(k, \ell, c) + c \cdot (t + 1) + f_{2.22}(k, \ell, t - 1, a - 1), c').\end{aligned}$$

Now let

$$S_C = Z_C \cup \bigcup_{C' \in \mathcal{C}_C} S'_{C'}.$$

Then, by Lemma 2.6,

$$\begin{aligned}\mathrm{td}_k(C, S_C) &\leqslant |Z_C| + \max_{C' \in \mathcal{C}_C} \mathrm{td}_k(C', S'_{C'}) \\ &\leqslant c \cdot (t + 1) + c \cdot f_{2.22}(k, \ell, t - 1, a - 1) + \\ &\quad + f_{2.22.1}(t, \ell' + 1, \\ &\qquad f_{2.23}(k, \ell, c) + c \cdot (t + 1) + f_{2.22}(k, \ell, t - 1, a - 1), c').\end{aligned}$$

Finally, let

$$S' = Z \cup \bigcup_{\substack{C \text{ connected component of} \\ \mathrm{torso}_{G'}(U', \mathcal{B}') - Z}} S_C.$$

Recall that $\mathrm{torso}_{G'}(U',\mathcal{B}') - Z = \mathrm{torso}_G(U,\mathcal{B}) - X - Z = \mathrm{torso}_G(U,\mathcal{B})[V_{\ell'}] - Z$. Then, by Lemma 2.6,

$$\begin{aligned}\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B})[V_{\ell'}],S') &\leqslant \mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B})[V_{\ell'}],Z) + \max_{\substack{C \text{ connected component}\\ \text{of } \mathrm{torso}_{G'}(U',\mathcal{B}')-Z}} \mathrm{td}_k(C,S_C)\\ &\leqslant f_{2.23}(k,\ell,c)\\ &\quad + c\cdot(t+1)\\ &\quad + c\cdot f_{2.22}(k,\ell,t-1,a-1)+\\ &\quad + f_{2.22.1}(t,\ell'+1,\\ &\qquad\qquad f_{2.23}(k,\ell,c)+c\cdot(t+1)+f_{2.22}(k,\ell,t-1,a-1),c')\\ &\leqslant f_{2.22.1}(t,\ell',c,c').\end{aligned}$$

Since $S \cap V_{\ell'} \subseteq S'$, this proves the claim. $\Diamond$

We now define $f_{2.22}(k,\ell,t,a)$ by

$$\begin{aligned}f_{2.22}(k,\ell,t,a) = \max\Big\{t,\ \tfrac{3}{2}(k-1),&\\ 2k-1+f_{2.22.1}\Big(\max\Big\{t,\tfrac{3}{2}(k-1)\Big\},1,k,\max\Big\{t,\tfrac{3}{2}(k-1)\Big\}\Big)\Big\}.&\end{aligned}$$

Let G be a graph that does not contain any k-ladder of length ℓ as a minor, let $(U,\mathcal{B})$ be a nice pair in G, and let $S \subseteq U$ be such that there is a tree decomposition of $(\mathrm{torso}_G(U,\mathcal{B}),S)$ of adhesion at most a and width less than t. To deduce the theorem, we will decompose $\mathrm{torso}_G(U,\mathcal{B})$ into nice pairs using Lemma 2.21. By Lemma 2.21 applied to $\mathrm{torso}_G(U,\mathcal{B})$ and S, there exists $S_0' \subseteq U$ containing S and $\mathcal{D} = (T,(W_x \mid x \in V(T)))$ a tree decomposition of $(\mathrm{torso}_G(U,\mathcal{B}),S_0')$ such that if $\mathcal{C}$ is the family of all the connected components of $\mathrm{torso}_G(U,\mathcal{B}) - S_0'$, then

2.21.(a) $\mathcal{D}$ has adhesion at most a;

2.21.(b) $\mathcal{D}$ has width less than $\max\{t,\frac{3}{2}(k-1)\}$;

2.21.(c) for every $x_1,x_2 \in V(T)$, for every $Z_1 \subseteq W_{x_1}$ and $Z_2 \subseteq W_{x_2}$ both of size k, either

- (i) there are k disjoint (Z_1,Z_2)-paths in G,
- (ii) there exists $z_1,z_2 \in E(T[x_1,x_2])$ with $|W_{z_1} \cap W_{z_2}| < k$, or
- (iii) $x_1 = x_2$, $|W_{x_1}| \leqslant \frac{3}{2}(k-1)$ and $|W_{x_1} \cap W_y| < k$ for every $y \in N_T(x_1)$; and

2.21.(d) for every $x_1x_2 \in E(T)$ with $|W_{x_1} \cap W_{x_2}| < k$, if there exists $x_3 \in N_T(x_2)$ such that $|W_{x_2} \cap W_{x_3}| \geqslant k$, then for every positive integer i, for every $Z_1,Z_2 \subseteq W_{x_1} \cap W_{x_2}$ both of size i, there are i disjoint (Z_1,Z_2)-paths in

$$\mathrm{torso}_G(U,\mathcal{B})\left[\bigcup_{z\in V(T_{x_2|x_1})} W_z \cup \bigcup_{C\in\mathcal{C}(x_2|x_1)} V(C)\right] \setminus \binom{W_{x_1}\cap W_{x_2}}{2},$$

where $\mathcal{C}(x_2 \mid x_1)$ is the family of all the connected components $C \in \mathcal{C}$ such that $N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \subseteq \bigcup_{z\in V(T_{x_2|x_1})} W_z$ and $N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \not\subseteq W_{x_1} \cap W_{x_2}$.

For every connected component $C \in \mathcal{C}$, fix $x_C \in V(T)$ such that $N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \subseteq W_{x_C}$.

Let $E_{<k}$ be the set of all the edges zz' of T such that $|W_z \cap W_{z'}| < k$. If $E_{<k} = E(T)$, then $\mathcal{D}$ has adhesion at most $k-1$ and so it is a k-dismantable tree decomposition of $(\mathrm{torso}_G(U,\mathcal{B}), S')$ of width less than $\max\{t, \frac{3}{2}(k-1)\}$. This implies that $\mathrm{td}_k(\mathrm{torso}_G(U,\mathcal{B}), S'_0) \leqslant \max\{t, \frac{3}{2}(k-1)\} \leqslant f_{2.22}(k,\ell,t,a)$.

Now suppose $E_{<k} \neq E(T)$. We root T on a vertex r which is incident to an edge in $E(T) \setminus E_{<k}$. Let $T_1, \ldots, T_m$ be all the connected components of $T \setminus E_{<k}$ that contain at least one edge.

Consider some $i \in [m]$. Let A_i be the adhesion between the root of T_i and its parent if $r \notin V(T_i)$, or the empty set if $r \in V(T_i)$. Let T'_i be the connected component containing T_i in $T - \bigcup_{j\in[m], j\neq i} V(T_j)$, see Figure 2.11. Then let T''_i be the subtree of T'_i rooted on the root of T_i. Let

$$U_i = \bigcup_{z \in V(T'_i)} \left(W_z \cup \bigcup_{C\in\mathcal{C}, N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \subseteq W_z} V(C) \right),$$

and let $\mathcal{B}_i$ be the family all of the sets

$$\bigcup_{z \in V(S)} \left(W_z \cup \bigcup_{\substack{C\in\mathcal{C}, N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \subseteq W_z, \\ N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \not\subseteq U_i}} V(C) \right)$$

for S connected component of $T - T'_i$.

Let

$$V_i = \bigcup_{z \in V(T''_i)} W_z \cup \bigcup_{C \in \mathcal{C}, x_C \in V(T''_i)} V(C)$$

and let $\mathcal{B}''_i$ be the family of all the sets

$$\bigcup_{z \in V(S)} \left(W_z \cup \bigcup_{\substack{C\in\mathcal{C}, N_{\mathrm{torso}_G(U,\mathcal{B})}(V(C)) \subseteq W_z, \\ x_C \notin V(T''_i)}} V(C) \right)$$

for S connected component of $T - T''_i$. Then let $G_i = \mathrm{torso}_{\mathrm{torso}_G(U,\mathcal{B})}(V_i, \mathcal{B}''_i)$. Observe that G can be obtained by $(<k)$-clique-sums of $G_1, \ldots, G_m$, and $V(G_i) \cap V(G_j) \subseteq S$ for every distinct $i, j \in [m]$. Hence it remains to bound $\mathrm{td}_k(G_i, S'_0 \cap V(G_i))$ for every $i \in [m]$.

By construction, $(U_i, \mathcal{B}_i)$ is a good pair in $\mathrm{torso}_G(U,\mathcal{B})$. Moreover, for every connected component S of $T - T'_i$, if the edge between T'_i and S is x_1x_2 with $x_1 \in V(T'_i)$, then $|W_{x_1} \cap W_{x_2}| < k$ and x_2 belongs to some T_j for $j \in [m] \setminus \{i\}$, which implies that x_2 has a neighbor x_3 with $|W_{x_2} \cap W_{x_3}| \geqslant k$. By 2.21.(d), this implies that for every positive integer i', for every $Z_1, Z_2 \subseteq W_{x_1} \cap W_{x_2}$ both of size i', there are i' disjoint (Z_1, Z_2)-paths in $\mathrm{torso}_G(U,\mathcal{B}) \left[\bigcup_{z\in V(T_{x_2|x_1})} W_z \cup \bigcup_{C\in\mathcal{C}(x_2\cup x_1)} V(C)\right] \setminus \binom{W_{x_1}\cap W_{x_2}}{2}$. This implies that $(U_i, \mathcal{B}_i)$ is a nice pair in $\mathrm{torso}_G(U,\mathcal{B})$. Now, since $(U,\mathcal{B})$ is a nice pair in G, we deduce by Lemma 2.18 that there exists $\mathcal{B}'_i$ such that $(U_i, \mathcal{B}'_i)$ is a nice pair in G and $\mathrm{torso}_G(U_i, \mathcal{B}'_i) = \mathrm{torso}_{\mathrm{torso}_G(U,\mathcal{B})}(U_i, \mathcal{B}_i)$. Now, observe that $G_i \subseteq \mathrm{torso}_G(U_i, \mathcal{B}'_i) \cup \binom{A_i}{2}$. Hence, to bound $\mathrm{td}_k(G_i, S'_0 \cap V(G_i))$, it is enough to bound the k-treedepth of $\mathrm{td}_k(\mathrm{torso}_G(U_i) \cup \binom{A_i}{2}, S'_0 \cap U_i)$. To do so, we will apply Claim 2.22.1.

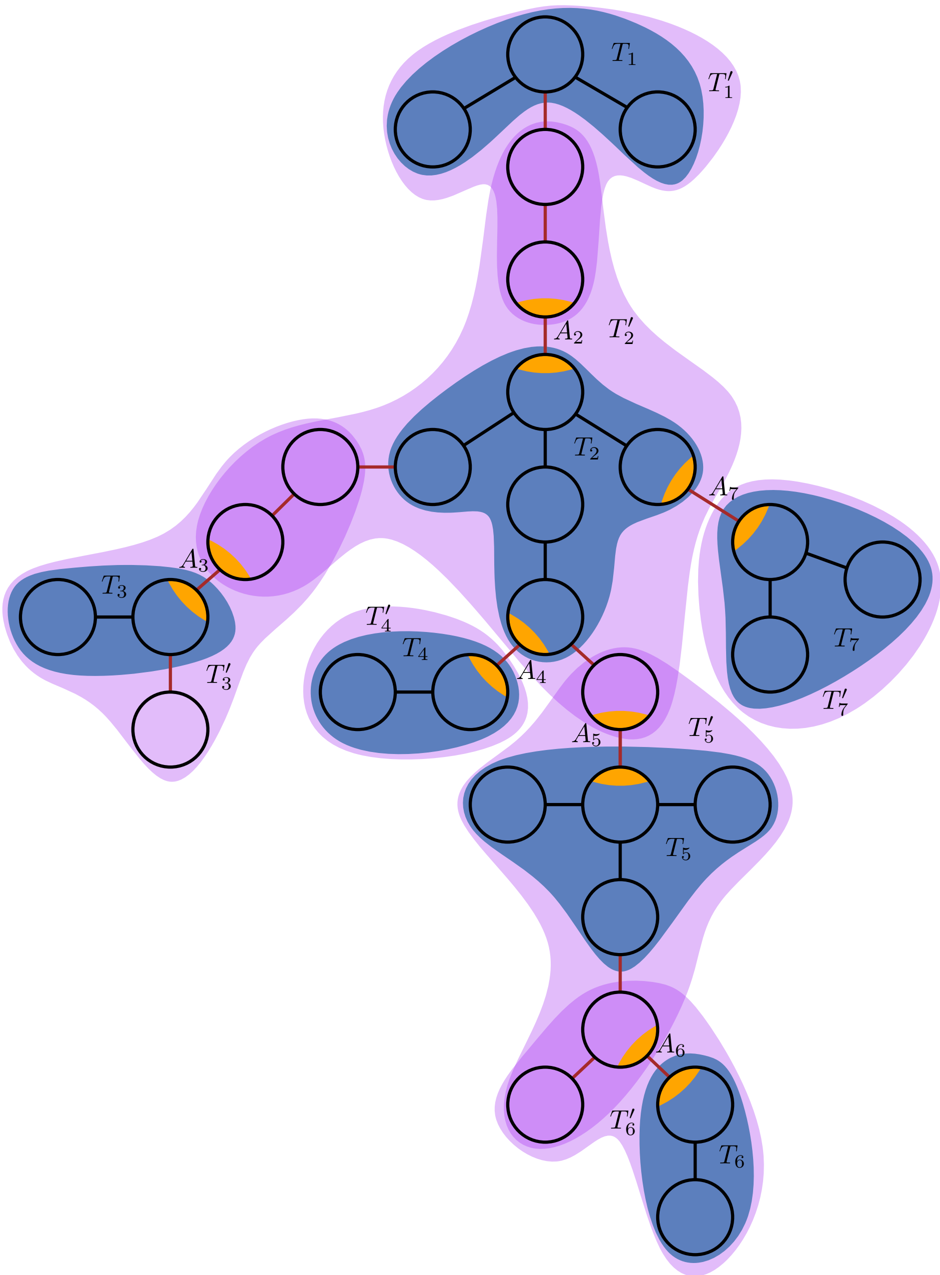


Figure 2.11: Application of Lemma 2.21 in the proof of Theorem 2.22. This yields a tree decomposition of $(\mathrm{torso}_G(U, \mathcal{B}), S_0')$ for some $S_0' \subseteq U$ containing S. For sake of clarity, the connected components of $\mathrm{torso}_G(U, \mathcal{B}) - S_0'$ are not depicted. We want to decompose $\mathrm{torso}_G(U, \mathcal{B})$ using this tree decomposition as a clique sum of some "nice" $G_1, \ldots, G_m$. To do so, we rely on the "well-connected" adhesions given by 2.21.(d). In orange are depicted the clique-sums we will perform. In black are depicted the edge $zz' \in E(T)$ with $|W_z \cap W_{z'}| \geqslant k$, and in brown those for which $|W_z \cap W_{z'}| < k$. By the properties of the tree decomposition, and in particular 2.21.(d), the sets $U_i = \bigcup_{z \in V(T_i')} W_z \cup \bigcup_{C \in \mathcal{C}, x_C \in V(T_i')} V(C)$ are in suitable nice pairs $(U_i, \mathcal{B}_i')$ in G, and so we can apply Claim 2.22.1 to decompose them.

Let V_0 be a subset of k vertices of the bag of the root of T_i. Observe that for every $W \subseteq U_i$, if there are no k disjoint (V_0, W)-paths in $\mathrm{torso}_G(U_i, \mathcal{B}'_i)$, then by Lemma 2.17 and 2.21.(c), $|W \cap W_z| < k$ for every $z \in V(T_i)$. This implies that $|W_z \cap W_{z'} \cap W| < k$ for every $zz' \in E(T'_i)$, and so $(T'_i, (W_z \cap W \mid z \in V(T'_i)))$ is a tree decomposition of $\left(\mathrm{torso}_G(U_i, \mathcal{B}'_i)[W], S'_0 \cap W\right)$ of adhesion at most $k-1$ and width less than $\max\left\{t, \frac{3}{2}(k-1)\right\}$. This proves that $\mathrm{td}_k\left(\mathrm{torso}_G(U_i, \mathcal{B}'_i)[W], S'_0 \cap W\right) \leqslant \max\left\{t, \frac{3}{2}(k-1)\right\}$. Hence, by Claim 2.22.1 applied to the path partition $(V_0, U_i \setminus V_0)$, $\ell' = 1$, $c = k$, $c' = \max\left\{t, \frac{3}{2}(k-1)\right\}$, $V_1 = U_i \setminus V_0$, there exists $S'_{0,i} \subseteq U_i \setminus V_0$ with $S'_0 \cap (U_i \setminus V_0) \subseteq S'_i$ such that

$$\mathrm{td}_k\left(\mathrm{torso}_G(U_i, \mathcal{B}'_i)[U_i \setminus V_0], S'_{0,i}\right) \leqslant f_{2.22.1}\left(\max\left\{t, \tfrac{3}{2}(k-1)\right\}, 1, k, \max\left\{t, \tfrac{3}{2}(k-1)\right\}\right).$$

Let $S'_i = (S'_{0,i} \cup V_0) \cap V(G_i)$. Then

$$\begin{aligned}
\mathrm{td}_k(G_i, S'_i) &\leqslant \mathrm{td}_k\left(\mathrm{torso}_G(U_i, \mathcal{B}'_i), S'_{0,i} \cup V_0\right) + |A_i| \\
&\leqslant |V_0| + \mathrm{td}_k\left(\mathrm{torso}_G(U_i, \mathcal{B}'_i)[U_i \setminus V_0], S'_{0,i}\right) + |A_i| \\
&\leqslant k + f_{2.22.1}\left(\max\left\{t, \tfrac{3}{2}(k-1)\right\}, 1, k, \max\left\{t, \tfrac{3}{2}(k-1)\right\}\right) + (k-1) \\
&\leqslant f_{2.22}(k, \ell, t, a).
\end{aligned}$$

Let $\mathcal{D}_i = (T_i, (W_z \mid z \in V(T_i)))$ be a k-dismantable tree decomposition of (G_i, S'_i) of width less than $f_{2.22}(k, \ell, t, a)$.

Finally, let

$$S' = \bigcup_{i \in [m]} S'_i.$$

Let $(i_1, j_1), \ldots, (i_{m-1}, j_{m-1})$ be all the pairs of distinct indices $i, j \in [m]$ such that there is an edge between $V(T'_j)$ and $V(T_i)$ in T, with the root of T_i being closer from r in T than the root of T_j. Note that for every $a \in [m-1]$, $A_{i_a} = V(G_{i_a}) \cap V(G_{j_a})$ induces a clique in both G_{i_a} and G_{j_a}, and $A_{i_a} \subseteq S'$. This implies that there exists $z_a \in V(T_{i_a})$ and $z'_a \in V(T_{j_a})$ such that $V(G_{i_a}) \cap V(G_{j_a}) \subseteq W^{i_a}_{z_a}, W^{j_a}_{z'_a}$. Now, let $T' = \left(\bigcup_{i \in [m]} T_i\right) \cup \{z_a z'_a \mid a \in [m-1]\}$ and $S' = \bigcup_{i \in [m]} (S'_i \cap V(G_i))$. Then, $S \subseteq S'_0 \subseteq S'$ and $(T', (W_x \mid x \in V(T')))$ is a k-dismantable tree decomposition of (G, S') of width less than $f_{2.22}(k, \ell, t, a)$. This proves the theorem. □

2.8 $k \times \ell$ grid minor of long $(2k-1)$-ladders

In this section, we show how to find a $k \times \ell$ grid in $T \square P_L$ for any tree T on $2k-1$ vertices, and some large enough integer L. Together with Theorem 1.20, this implies Corollary 1.21.

Lemma 2.24. *Let k, ℓ be positive integers. For every tree T on $2k-1$ vertices, there exists an integer L such that the $k \times \ell$ grid is a minor of $T \square P_L$.*

First, let us introduce some notation. Given two graphs H, G, we write $H \sqsubseteq G$ if there is a sequence $\varphi_1, \ldots, \varphi_m$ of injective functions from $V(H)$ to $V(G)$ such that

(i) for every $i \in [m-1]$, φ_i and φ_{i+1} differs on exactly one vertex $x \in V(H)$, and $\varphi_i(x)\varphi_{i+1}(x)$ is an edge in G; and

(ii) for every $xy \in E(H)$, there exists $i \in [m]$ such that $\varphi_i(x)\varphi_i(y) \in E(G)$.

Less formally, we can see the vertices of H as tokens placed on the vertices of G, and we successively move these tokens along the edges of G in such a way that every edge of H is such that the corresponding tokens are adjacent in G at least once in this sequence. See Figure 2.12.

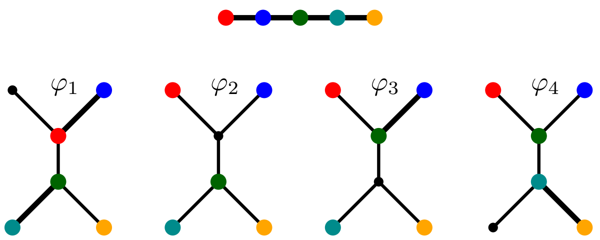


Figure 2.12: A sequence $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ witnessing that $P_5 \sqsubseteq T$, where T is the tree on 6 vertices having two internal vertices of degree 3. Informally, we place the vertices of P_5 as token on the graph T, and we move them one by one in such a way that for every edge xy of P_5, the tokens of x and y are adjacent in T at least once. The fat edges represent these realizations $\varphi_i(x)\varphi_i(y) \in E(T)$ of the edges $xy \in E(P_5)$.

This definition is motivated by the following observation.

Observation 2.25. *Let G, H be two graphs. If $H \sqsubseteq G$, then for every positive integer ℓ, there exists a positive integer L such that $H \square P_\ell$ is a minor of $G \square P_L$.*

Proof. Since $H \sqsubseteq G$, there is a sequence $\varphi_1, \ldots, \varphi_m$ of injective functions from $V(H)$ to $V(G)$ such that

(i) for every $i \in [m-1]$, φ_i and φ_{i+1} differs on exactly one vertex $x \in V(H)$, and $\varphi_i(x)\varphi_{i+1}(x)$ are adjacent in G; and

(ii) for every $xy \in E(H)$, there exists $i \in [m]$ such that $\varphi_i(x)\varphi_i(y) \in E(G)$.

Let $L = \ell \cdot (2m-1)$. We label the vertices of $G \square P_L$ by (x, i) for $x \in V(G)$ and $i \in [L]$. For every $x \in V(H)$, let $B_x^0 = \{(\varphi_i(x), i) \mid i \in [m]\} \cup \{(\varphi_i(x), i+1) \mid i \in [m-1], \varphi_i(x) \neq \varphi_{i+1}(x)\}$. Then let $B_x^1 = B_x^0 \cup \{(u, 2m-i) \mid (u, i) \in B_x^0\}$, and finally $B_{(x,j)} = \{(u, i + (2m-1)(j-1)) \mid (u, i) \in B_x^1\}$ for every $j \in [\ell]$. See Figure 2.13. It is not hard to check that $(G \square P_L)[B_{x,j}]$ is connected, that the sets $(B_{x,j})_{x \in V(H), j \in [\ell]}$ are pairwise disjoint, and for every $xy \in E(H)$, there is an edge between $B_{(x,j)}$ and $B_{(y,j)}$ in $G \square P_L$, for every $j \in [\ell]$. Moreover, there is an edge between $B_{(x,j)}$ and $B_{(x,j+1)}$ in $G \square P_L$ for every $j \in [\ell - 1]$. Hence $(B_{(x,j)} \mid (x, j) \in V(H \square P_\ell))$ is a model of $H \square P_\ell$ in $G \square P_L$. This proves the observation. □

Using Observation 2.25, it is enough to show the following to prove Lemma 2.24.

Lemma 2.26. *For every positive integer k, for every tree T on $2k-1$ vertices, $P_k \sqsubseteq T$.*

Proof. Let $P = (u_0, \ldots, u_p)$ be a longest path in T. If $p \geqslant k-1$, then set $m = 1$ and $\varphi_1(i) = u_{i-1}$, for every every vertex i of the path $P_k = (1, \ldots, k)$, and it follows that $P_k \sqsubseteq T$. Now we assume that $p \leqslant k-2$, and so $|V(T) \setminus V(P)| \geqslant k$. Let φ_0 be any injection from $[k] = V(P_k)$ to $V(T) \setminus V(P)$. We claim that it is enough to show that for every distinct $x, y \in [k] = V(P_k)$, there is a sequence $(\varphi_1^{x,y}, \ldots, \varphi_{m_{x,y}}^{x,y})$ of injections from $[k]$ to $V(T)$ such that

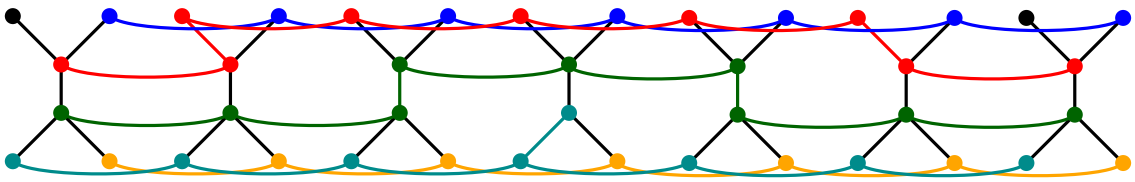


Figure 2.13: Illustration for Observation 2.25. Using the sequence $(\varphi_1, \dots, \varphi_4)$ given in Figure 2.12, we construct a model of $P_5 \square P_\ell$ in $T \square P_{7\cdot\ell}$ by gluing together ℓ copies of the model $(B_x^1 \mid x \in V(P_5))$ depicted here. For sake of clarity, some edges of $T \square P_7$ which are not used by this model are not depicted.

(i) for every $i \in [m_{x,y} - 1]$, $\varphi_i^{x,y}$ and $\varphi_{i+1}^{x,y}$ differs on exactly one vertex $x \in V(H)$, and $\varphi_i^{x,y}(x)\varphi_{i+1}^{x,y}(x)$ are adjacent in G; and

(ii) $\varphi_{m_{x,y}}^{x,y}(x)\varphi_{m_{x,y}}^{x,y}(y) \in E(T)$;

(iii) $\varphi_1^{x,y} = \varphi_0$.

Indeed, the sequence $\varphi_1^{1,2}, \dots, \varphi_{m_{1,2}-1}^{1,2}, \varphi_{m_{1,2}}^{1,2}, \varphi_{m_{1,2}-1}^{1,2}, \dots, \varphi_2^{1,2}, \varphi_1^{2,3}, \dots, \dots, \varphi_{m_{(k-1),k}}^{(k-1),k}$ then witnesses that $P_k \sqsubseteq T$. Note that this argument also yields $K_k \sqsubseteq T$.

For sake of clarity, we fix $x, y \in [k] = V(P_k)$ distinct, and we will write φ_i for $\varphi_i^{x,y}$. First set $\varphi_1 = \varphi_0$. Let u_i (resp. u_j) be the vertex of P closest to x (resp. y). Without loss of generality, assume that $i \leqslant j$. By maximality of P, $|V(P[u_0, u_i[)| \geqslant |V(T[\varphi_0(x), u_i[)|$ and $|V(P]u_j, u_p])| \geqslant |V(T[\varphi_0(y), u_j[)|$.

Then we move the vertices of $\varphi_0([k]) \cap V(T[\varphi_0(x), u_i[)$ into $\{u_0, \dots, u_{i-1}\}$ by a sequence $\varphi_1, \dots, \varphi_{m_0}$ satisfying (i) and such that $\varphi_{m_0}(x) = u_{i-1}$, and the vertices not in $T[\varphi_0(x), u_i[$ are not moved. Then, we move the vertices of $\varphi_0([k]) \cap V(T[\varphi_0(y), u_j[)$ into $\{u_{j+1}, \dots, u_p\}$ by a sequence $\varphi_{m_0+1}, \dots, \varphi_{m_1}$ satisfying (i) and such that $\varphi_{m_1}(y) = u_{j+1}$, and the vertices not in $T[\varphi_0(y), u_j[$ are not moved. Finally, we move y from u_{j+1} to u_i by a sequence $\varphi_{m_1+1}, \dots, \varphi_m$ satisfying (i) and such that $\varphi_i(z) = \varphi_{m_1}(z)$ for every $z \in [k] \setminus \{y\}$. This sequence is depicted on Figure 2.14. Then $\varphi_m(x)\varphi_m(y) = u_{j-1}u_j$ is an edge of T, and so $\varphi_1, \dots, \varphi_m$ is as claimed. This proves the lemma. □

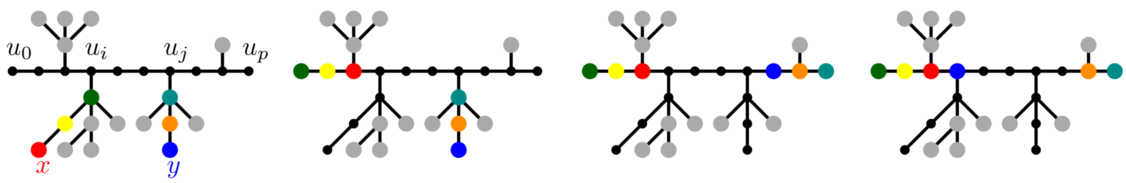


Figure 2.14: The main steps in the sequence given in the proof of Lemma 2.26 to make adjacent the red token x and the blue token y. In gray are depicted the tokens which are not moved.

PART II

Excluding a rooted minor

CHAPTER 3

Rooted and rich minors

This chapter contains joint work with Jędrzej Hodor, Hoang La, and Piotr Micek, and is an introduction to the technique used in [HLMR24a, HLMR24b, HLMR25].

3.1 Motivation

The purpose of this chapter is to present the main tool used throughout this part. This technique is an attempt to answer to the general question:

Question. *Let X be a graph and let $x \in V(X)$. Knowing a structure for $(X - x)$-minor-free graphs, can we deduce a structure for X-minor-free graphs?*

To illustrate this problem, consider the following concrete example.

Locally-bounded vertex cover.

Question. *For which minor-closed class of graphs $\mathcal{C}$ there is a function f such that for every $G \in \mathcal{G}$,*

$$\mathrm{vc}(G) \leqslant f(\mathrm{radius}(G))?$$

We say that such a class $\mathcal{C}$ has *locally-bounded vertex cover*. A first step to answer this question is to find “obstructions”, that is a minimal family of graphs that do not have this property. Here, a good candidate is the following family. For every integer ℓ with $\ell \geqslant 2$, let X_ℓ be the graph obtained from a matching M_ℓ on ℓ edges by adding a vertex u adjacent to exactly one vertex in each edge of M_ℓ. In other words, X_ℓ is obtained from the star with ℓ leaves by subdividing every edge once. See Figure 3.1.

The point of this construction is that X_ℓ has radius 2, but it has a matching on ℓ edges, which implies $\mathrm{vc}(X_\ell) \geqslant \ell$. Hence, if a minor-closed class of graphs has locally-bounded vertex cover, then there exists an integer ℓ with $\ell \geqslant 2$ such that $X_\ell \notin \mathcal{C}$. We will show that this condition is also sufficient.

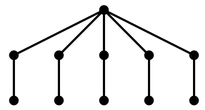


Figure 3.1: The graph X_5.

Theorem 3.1. *Let ℓ be an integer with $\ell \geqslant 2$. For every graph G, if X_ℓ is not a minor of G, then*

$$\mathrm{vc}(G) \leqslant (2\ell - 1)\,\mathrm{radius}(G).$$

To prove it, we will use the following crucial observation.

Observation 3.2. *Let ℓ be an integer with $\ell \geqslant 2$, let G be a graph, and let $u \in V(G)$. If there is a matching M on ℓ edges in $G - u$ such that every edge in M intersects $N_G(u)$, then X_ℓ is a minor of G.*

The point of this observation is that it shifts the problem of excluding a subdivided star, to the problem of excluding a matching, which is much simpler.

Let G be a graph, and let $S \subseteq V(G)$. An *S-matching* in G is a set of pairwise disjoint edges in G, all incident to at least one vertex in S. See Figure 3.2. A *vertex cover* of (G, S) is a set

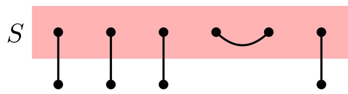


Figure 3.2: An S-matching.

$X \subseteq V(G)$ intersecting every edge intersecting S, and we denote by $\mathrm{vc}(G, S)$ the minimum order of a vertex cover of (G, S).

First, observe that if M is an S-matching in G, then $\mathrm{vc}(G, S) \geqslant |M|$. Indeed, any vertex cover of (G, S) must intersect every edge in M. The following lemma shows that reciprocally, if there is no large S-matching in G, then $\mathrm{vc}(G, S)$ is bounded.

Lemma 3.3. *Let ℓ be an integer with $\ell \geqslant 2$, let G be a graph, and let $S \subseteq V(G)$. If there is no S-matching of order ℓ in G, then*

$$\mathrm{vc}(G, S) \leqslant 2\ell - 2.$$

Proof. Let M be an inclusion-wise maximal S-matching in G. By maximality of M, $\bigcup_{uv \in M} \{u, v\}$ is a vertex cover of (G, S), which has size $2|M| \leqslant 2\ell - 2$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let G be a graph that does no contain X_ℓ as a minor. We proceed by induction on $\mathrm{radius}(G)$. If $\mathrm{radius}(G) = 0$, then G has only one vertex and clearly $\mathrm{vc}(G) = 0$. Now suppose $\mathrm{radius}(G) > 0$ and that the result holds for graph of smaller radius.

Let $u \in V(G)$ be a vertex at distance at most $\mathrm{radius}(G)$ from every other vertices in G. By Observation 3.2, there is no $N_G(u)$-matching of order ℓ in $G - u$. Hence, by Lemma 3.3, there is a vertex cover $X_0 \subseteq V(G - u)$ of $(G - u, N_G(u))$ of size at most $2\ell - 2$. Take such an X_0 which is inclusion-wise minimal. This implies that every vertex in X_0 is at distance at most 2 from u in

G, and so $G[\{u\} \cup S \cup X_0]$ is connected. Let G' be the graph obtained from G by contracting $\{u\} \cup S \cup X_0$ into a single vertex u'.

Observe that u' is at distance at most $\mathrm{radius}(G) - 1$ from every other vertex in G'. Hence $\mathrm{radius}(G') \leqslant \mathrm{radius}(G) - 1$. Moreover, G' is a minor of G, and so X_ℓ is not a minor of G'. By the induction hypothesis, this implies that

$$\mathrm{vc}(G') \leqslant (2\ell - 1)(\mathrm{radius}(G) - 1).$$

Let X' be a vertex cover of G' of size at most $(2\ell - 1)(\mathrm{radius}(G) - 1)$. Finally, let

$$X = (X' \setminus \{u'\}) \cup X_0 \cup \{u\}.$$

See Figure 3.3. First, by construction, $|X| \leqslant (2\ell - 1)\,\mathrm{radius}(G)$. It remains to show that X is a vertex cover of G. Let vw be an edge in G. We want to show that $v \in X$ or $w \in X$. If $v \in \{u\} \cup X_0$ or $w \in \{u\} \cup X_0$, then we are done. Now suppose that $v, w \notin \{u\} \cup X_0$. If $v \in N_G(u)$ or $w \in N_G(u)$, then, since X_0 is a vertex cover of $(G - u, N(u))$, either $v \in X_0$ or $w \in X_0$ and we are done. Otherwise, $v, w \notin \{u\} \cup N(u) \cup X_0$. Hence vw is an edge of G', and so $v \in X'$ or $w \in X'$. But since v and w are different from u', this implies that $v \in X$ or $w \in X$, which concludes the proof of the theorem. □

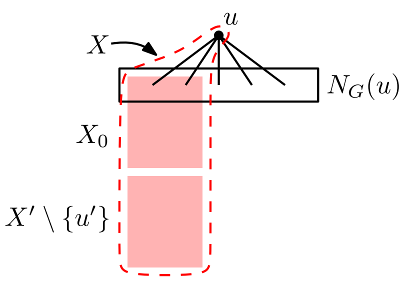


Figure 3.3: Illustration for the proof of Theorem 3.1.

Note that a slightly better analysis in the proof would yield $\mathrm{vc}(G) \leqslant (2\ell - 2)\,\mathrm{radius}(G)$.

3.2 Rooted minors and decompositions focused on a subset of vertices

In this section, we describe a general framework to generalize the method used in the previous section. This technique will be the subject of Chapter 4. First, we need a rooted version of minors, to generalize the concept of rooted matchings.

Let G, X be two graphs, let $S \subseteq V(G)$, and let $U \subseteq V(X)$. An *S-rooted model of* (X, U) in G is a family $(B_x \mid x \in V(X))$ of pairwise disjoint nonempty subsets of $V(G)$ such that

(i) for every $x \in V(X)$, $G[B_x]$ is connected,

(ii) for every $x \in U$, $S \cap B_x \neq \emptyset$, and

(iii) for every edge $xy \in E(X)$, there is an edge between B_x and B_y in G.

For example, if M is the matching on ℓ edges, and if $U \subseteq V(M)$ contains exactly one vertex per edge of M, then G contains an S-rooted model of (M, U) if and only if there is an S-matching of order ℓ in G. When $U = V(X)$, we simply say that $(B_x \mid x \in V(X))$ is an *S-rooted model of X*. This will be the case in most applications, and so we will suppose this is the case in this overview. We say that X is an *S-rooted minor* of G if there is an S-rooted model of X in G.

The point of this definition is the following generalization of Observation 3.2.

Observation 3.4. *Let X be a graph and let $x \in V(X)$, let G be a graph and let $u \in V(G)$. If $G - u$ contains an $N_G(u)$-rooted model of $X - x$, then X is a minor of G.*

Therefore, to prove a structural property on X-minor-free graphs, we will prove a structure for graphs excluding $X - x$ as a rooted minor, and then apply it iteratively as in the proof of Theorem 3.1. In the remaining of this section, we describe some of the local structures we can obtain when a rooted minor is excluded, and their applications. The proofs will be given in Chapter 4.

Rooted paths and treedepth. As mentioned in the introduction, a class of graphs has bounded treedepth if and only if it excludes a path as a minor. This can be generalized to the context of rooted minors through the following notion of treedepth focused on a subset of vertices.

First, we need a few definitions. A *rooted forest* is a disjoint union of rooted trees. The *vertex-height* of a rooted forest F is the maximum number of vertices on a path from a root to a leaf in F, and the *depth* of a vertex $u \in V(F)$ is the number of vertices in the path between u and the root of its component. For two vertices u, v in a rooted forest F, we say that u is a *descendant* of v in F if v lies on the path from a root to u in F. The *closure* of F is the graph with vertex set $V(F)$ and edge set $\{vw \mid v \neq w$ and v is a descendant of w in $F\}$. We say that F is an *elimination forest* of G if $V(F) = V(G)$ and G is a subgraph of the closure of F. It is a classical observation that the *treedepth* of a graph G, is 0 if G is empty, and otherwise is the minimum vertex-height of an elimination forest of G. When F is a non rooted forest, the vertex-height of F is the minimum vertex-height of a rooted forest with underlying non rooted forest F.

Let G be a graph and let $S \subseteq V(G)$. An *elimination forest of* (G, S) is a rooted forest F with vertex set S such that

(ef1) for every edge uv of $G[S]$, either u is an ancestor of v in F, or v is an ancestor of u in F,

(ef2) for every component C of $G - S$, there exists a root-to-leaf path P in F such that $N_G(V(C)) \subseteq V(P)$.

See Figure 3.4. The *treedepth of* (G, S), denoted by $\mathrm{td}(G, S)$, is the minimum height of an elimination forest of (G, S). Note that this is consistent with the definition given in Chapter 2. Note that $S \mapsto \mathrm{td}(G, S)$ is not necessarily monotone for the inclusion. For example, when G is a star with ℓ leaves, S_1 is the set of leaves and $S_2 = V(G)$, we have $S_1 \subseteq S_2$ with $\mathrm{td}(G, S_1) = \ell$ while $\mathrm{td}(G, S_2) = 2$. For this reason, it will be convenient to consider the parameter $\overline{\mathrm{td}}(\cdot, \cdot)$, defined by

$$\overline{\mathrm{td}}(G, S) = \min_{S \subseteq S' \subseteq V(G)} \mathrm{td}(G, S').$$

The S-rooted counterpart of the fact that paths have unbounded treedepth is that if G has an S-rooted model of a long path, then the treedepth of (G, S') is large for every superset S' of S.

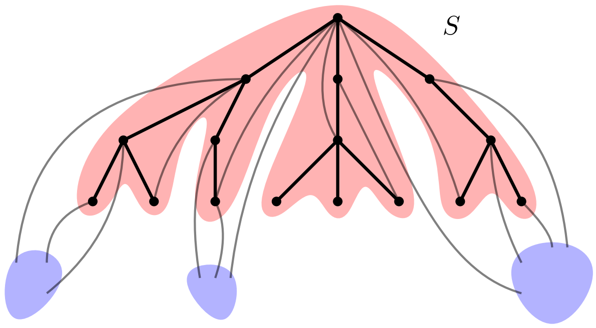


Figure 3.4: An elimination forest of (G, S). The set S is depicted in red, and the components of $G - S$ are depicted in blue.

Lemma 3.5. *Let k be a nonnegative integer, let G be a graph and let $S \subseteq V(G)$. If P_{2^k} is an S-rooted minor of G, then*

$$\overline{\mathrm{td}}(G, S) > k.$$

Proof. We proceed by induction on k. The result is clear if $k = 0$. Now suppose $k > 0$ and that the result holds for $k - 1$.

First, observe that we can assume that G is connected. Let $S' \subseteq V(G)$ containing S and let F be an elimination forest of (G, S') of minimum vertex-height. Since G is connected, this forest is connected. Let r be the root of F. Observe that $\mathrm{td}(G, S') \geqslant 1 + \mathrm{td}(G - r, S' \setminus \{r\})$. But, on the other hand, $G - r$ contains an $S \setminus \{r\}$-rooted model of $P_{2^{k-1}}$. Hence, by the induction hypothesis,

$$\mathrm{td}(G, S') \geqslant 1 + \mathrm{td}(G - r, S' \setminus \{r\}) > 1 + (k - 1) = k. \qquad \square$$

In Section 4.2, we will show that reciprocally, if there is no S-rooted model of the path on ℓ vertices in G, then $\overline{\mathrm{td}}(G, S)$ is bounded by a function of ℓ.

Theorem 3.6. *Let ℓ be a positive integer, let G be a graph, and let $S \subseteq V(G)$. If P_ℓ is not an S-rooted minor of G, then*

$$\overline{\mathrm{td}}(G, S) \leqslant \binom{\ell}{2}.$$

Rooted forests and pathwidth. As mentioned in the introduction, a class of graphs has bounded pathwidth if and only if it excludes a tree as a minor. This can be generalized to the context of rooted minors through the following notion of pathwidth focused on a subset of vertices.

Let G be a graph and let $S \subseteq V(G)$. A *path decomposition of* (G, S) is a sequence $(W_1, \dots, W_m)$ of subsets of S such that

(pd1) for every $u \in S$, $\{i \in [m] \mid u \in W_i\}$ is a nonempty interval of integers,

(pd2) for every edge $uv \in E(G[S])$, there exists $i \in [m]$ such that $u, v \in W_i$, and

(pd3) for every component C of $G - S$, there exists $i \in [m]$ such that $N_G(V(C)) \subseteq W_i$.

See Figure 3.5. We define the *pathwidth*, denoted by $\mathrm{pw}(G, S)$, as the minimum width of a path decomposition of (G, S). Moreover, we denote by $\overline{\mathrm{pw}}(G, S)$ the minimum of $\mathrm{pw}(G, S')$ over all $S' \subseteq V(G)$ containing S.

Let ℓ be a positive integer. We denote by T_ℓ the complete ternary tree of vertex-height ℓ.

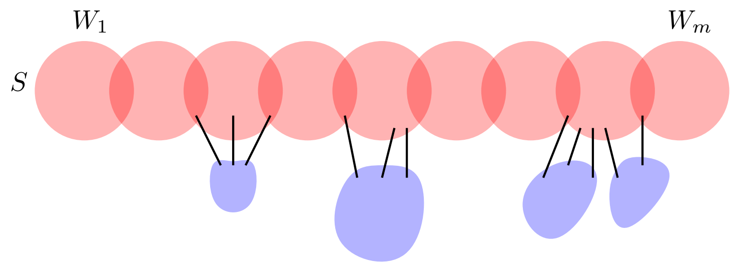


Figure 3.5: A path decomposition $(W_1, \dots, W_m)$ of (G, S). The set $S = \bigcup_{i\in[m]} W_i$ is depicted in red, and the components of $G - S$ are depicted in blue.

Lemma 3.7. *Let ℓ be a positive integer, let G be a graph, and let $S \subseteq V(G)$. If T_ℓ is an S-rooted minor of G, then*

$$\overline{\mathrm{pw}}(G, S) \geqslant \ell - 1.$$

Proof. We proceed by induction on ℓ. The result is clear if $\ell = 0$. Now suppose $\ell > 0$ and that the result holds for $\ell - 1$.

First, observe that we can assume that G is connected. Let $\mathcal{M}$ be an S-rooted model of T_ℓ. Let $S' \subseteq V(G)$ containing S and let $(W_1, \dots, W_m)$ be a path decomposition of (G, S') of minimum width. By taking S inclusion-wise minimal for containing $\mathcal{M}$, we can assume that W_1 and W_m both intersect a branch set of $\mathcal{M}$. Let P be a path in G between these two branch sets corresponding to a path in T_k between the aforementioned branch sets. Observe that $V(P)$ intersects W_i for every $i \in [m]$. Hence, $\mathrm{pw}(G - V(P), S' \setminus V(P)) < \mathrm{pw}(G, S')$. But on the other hand, $G - V(P)$ contains $F_{\ell-1}$ as an S-rooted minor. Hence, by the induction hypothesis,

$$\mathrm{pw}(G, S') > \mathrm{pw}(G - V(P), S' \setminus V(P)) \geqslant \ell - 1. \qquad \square$$

We will prove in Section 4.3, that reciprocally, if T_k is not an S-rooted minor of G, then $\overline{\mathrm{pw}}(G, S)$ is bounded.

Theorem 3.8. *Let F be a forest with at least two vertices, let G be a graph, and let $S \subseteq V(G)$. If F is not an S-rooted minor of G, then*

$$\overline{\mathrm{pw}}(G, S') \leqslant 2|V(F)| - 3.$$

Outer-rooted grids and treewidth. For treewidth, the natural notion of treewidth focused on a subset of vertices is the following. Let G be a graph and let $S \subseteq V(G)$. A *tree decomposition of* (G, S) is a pair $(T, (W_x \mid x \in V(T)))$ where T is a tree and $W_x \subseteq S$ for every $X \in V(T)$ such that

(td1) for every $u \in S$, $\{x \in V(T) \mid u \in W_x\}$ induces a nonempty connected subtree of T, and

(td2) for every connected component C of $G - S$, there exists $x \in V(T)$ such that $N_G(V(C)) \subseteq W_x$.

See Figure 3.6. The width of this tree decomposition is $\max_{x\in V(T)} |W_x| - 1$. The sets W_x, for $x \in V(T)$, are called the *bags* of this tree decomposition. The *treewidth of* (G, S), denoted by

$\mathrm{tw}(G,S)$, is the minimum width of a tree decomposition of (G,S). Again, we define an inclusion-monotone version by

$$\overline{\mathrm{tw}}(G,S) = \min_{S\subseteq S'\subseteq V(G)} \mathrm{tw}(G,S').$$

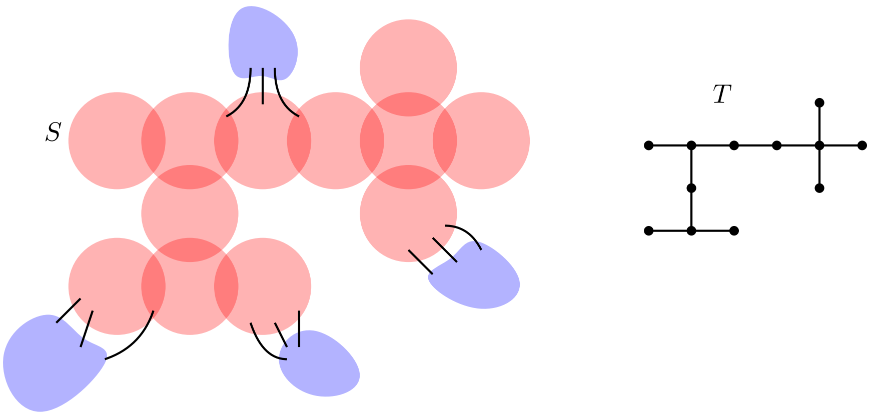


Figure 3.6: A tree decomposition $(T,(W_x \mid x \in V(T)))$ of (G,S). In blue are depicted the components of $G - S$.

As mentioned in the introduction, the Grid-Minor Theorem of Robertson and Seymour (Theorem 1.10) asserts that a class of graphs has bounded treewidth if and only if it excludes a grid as a minor. However, it is not true that if there is no S-rooted model of a fixed grid in G, then there exists $S' \subseteq V(G)$ containing S with $\mathrm{tw}(G,S')$ bounded.

Lemma 3.9. *Let ℓ be a positive integer, let X be the $\ell\times\ell$ grid and let U be the vertex set of the outer face of X, let G be a graph, and let $S \subseteq V(G)$. If there is an S-rooted model of (X,U) in G, then*

$$\overline{\mathrm{tw}}(G,S) \geqslant \ell.$$

The proof will use the following variant of Lemma 1.17 focused on a subset of vertices.

Lemma 3.10. *Let G be a graph, let $S \subseteq V(G)$, and let $\mathcal{D}$ be a tree decomposition of (G,S). For every positive integer d, for every family $\mathcal{F}$ of connected subgraphs of G, if every member of $\mathcal{F}$ intersects S, then either*

(1) *there are d pairwise disjoint members of $\mathcal{F}$, or*

(2) *there is a set $Z \subseteq S$ which is the union of at most $d-1$ bags of $\mathcal{D}$ such that Z intersects every member of $\mathcal{F}$.*

Proof. Let G' be the graph with vertex set S and edges all the pairs uv of distinct vertices in S such that either $uv \in E(G)$, or there exists a component C of $G-S$ such that $u,v \in N_G(V(C))$. Observe that $\mathcal{D}$ is then a tree decomposition of G'.

Let $\mathcal{F}'$ be the family of all the subgraphs of G' of the form $\pi(F) = G'[V(F)]$ for $F \in \mathcal{F}$. It follows from the definition of G' that every member of $\mathcal{F}'$ is a connected subgraph of G'. Moreover, for every $F_1,F_2 \in \mathcal{F}$, if $V(F_1)\cap V(F_2) \neq \emptyset$, then there is a bag in $\mathcal{D}$ intersecting both $V(F_1)$ and $V(F_2)$, and so the projections of $\pi(F_1)$ and $\pi(F_2)$ in $\mathcal{D}$ have a nonempty intersection. Hence, by Lemma 1.17, either

(i) there are d members of $\mathcal{F}'$ whose projections on $\mathcal{D}$ are pairwise disjoint, and so the corresponding members of $\mathcal{F}$ are pairwise disjoint, or

(ii) there is a set Z which is the union of at most $d-1$ bags of $\mathcal{D}$ intersecting every member of $\mathcal{F}'$, and so every member of $\mathcal{F}$. □

Lemma 3.9 is now a simple consequence of Lemma 3.10.

Proof of Lemma 3.9. Let $(B_x \mid x \in V(X))$ be an S-rooted model of (X, U) in G. Recall that the vertex set of X is $[\ell] \times [\ell]$. For every $i, j \in [\ell]$, let

$$V_{i,j} = \bigcup_{j' \in [\ell]} B_{(i,j')} \cup \bigcup_{i' \in [\ell]} B_{(i',j)}.$$

Then the family $\mathcal{F} = \{G[V_{i,j}] \mid i, j \in [\ell]\}$ has no two disjoint members. Let $S' \subseteq V(G)$ containing S. By Lemma 3.10, there exists $Z \subseteq V(G)$ of size at most $\mathrm{tw}(G, S')$ intersecting every member of $\mathcal{F}$. By the definition of $\mathcal{F}$, this implies $|Z| \geqslant \ell$, and so

$$\mathrm{tw}(G, S') \geqslant |Z| \geqslant \ell.$$ □

We will prove in Section 4.4 that it is actually the only obstruction to have bounded $\overline{\mathrm{tw}}(G, S)$. The proof is a direct application of a result of [MSW17].

Theorem 3.11. *There is a function $f_{3.11} \colon \mathbb{N} \to \mathbb{N}$ such that the following holds. Let ℓ be an integer, let X be the $\ell \times \ell$ grid and let U be the vertex set of the outer face of X, let G be a graph, and let $S \subseteq V(G)$. If there is no S-rooted model of (X, U) in G, then*

$$\overline{\mathrm{tw}}(G, S) \leqslant f_{3.11}(\ell).$$

3.3 Rich minors and hitting sets

The technique presented in the previous section shows how to deduce a structure for X-minor-free graphs knowing a structure for $(X - x)$-minor-free graphs for $x \in V(X)$. However, to go further and adding more than one vertex to the excluded minor, we need a more general notion of rooted minors that will enable us to set up inductions on the structure of the excluded minor.

Let G, X be two graphs, let $\mathcal{F}$ be a family of connected subgraphs of G. An *$\mathcal{F}$-rich model of X* in G is a family $(B_x \mid x \in V(X))$ of pairwise disjoint nonempty subsets of $V(G)$ such that

(i) for every $x \in V(X)$, $G[B_x]$ is connected,

(ii) for every $x \in V(X)$, there exists $F \in \mathcal{F}$ such that $V(F) \subseteq B_x$, and

(iii) for every edge $xy \in E(X)$, there is an edge between B_x and B_y in G.

For example, if $S \subseteq V(G)$ and $\mathcal{F}$ is the family $\{G[\{u\}] \mid u \in S\}$, then an $\mathcal{F}$-rich model of X is just an S-rooted model of X as defined in the previous section.

An important remark is that we will need some global structure on G to obtain a meaningful structural property if G has no $\mathcal{F}$-rich model of X. Indeed, we need to rule out the case $G = K_n$ and $\mathcal{F}$ is the set of all the subgraphs of K_n with more than $\frac{n}{2}$ vertices. In this case, there are no two disjoint members of $\mathcal{F}$, and so as long as X has an edge, G has no $\mathcal{F}$-rich model of X. To solve this issue, we will suppose that G has bounded treewidth, or at least that G excludes K_t as a minor for a fixed integer t, which will be the case in our applications. In this overview, we will consider only the bounded treewidth case. We now give examples of structures for graphs with no $\mathcal{F}$-rich model of X.

Excluding a stable set. Let ℓ be a positive integer. We denote by $\overline{K_\ell}$ the graph with ℓ vertices and no edges. Let G be a graph and let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of $\overline{K_\ell}$. In other words, there are no ℓ pairwise disjoint members of $\mathcal{F}$. Here, the relevant result is the already mentioned Lemma 1.17. As a consequence, we obtain the following property when G has bounded treewidth.

Corollary 3.12. *Let ℓ, t be positive integers, let G be a graph with $\mathrm{tw}(G) < t$, and let $\mathcal{F}$ be a family of connected subgraphs of G. If G has no $\mathcal{F}$-rich model of $\overline{K_\ell}$, then there exists $S \subseteq V(G)$ such that*

(A) $V(F) \cap S \neq \emptyset$ *for every* $F \in \mathcal{F}$, *and*

(B) $|S| \leqslant t(\ell - 1)$.

We will see that this is a rather general phenomenon: if G has no $\mathcal{F}$-rich model of a graph X, then there is a *hitting set* S of $\mathcal{F}$, that is a set of vertices intersecting every member of $\mathcal{F}$, with a specific structure depending on the structure of X.

Excluding a star. Let G be a connected graph that does not contains the star $K_{1,\ell}$ as a minor. Let $u \in V(G)$, and for every nonnegative integer i, let P_i be the set of vertices of G at distance i from u. The crucial observation is that $G[P_0 \cup \cdots \cup P_{i-1}]$ is connected for every nonnegative integer i, and every vertex in P_i has a neighbor in P_{i-1}. Hence, contracting $P_0 \cup \cdots \cup P_{i-1}$, we have that $K_{1,|P_i|}$ is a minor of G, and so $|P_i| \leqslant \ell - 1$. Therefore, the sequence $(P_0, \dots, P_m)$ is a partition of $V(G)$ such that every edge of G is either within one part or between two consecutive parts, and each part has bounded size. It turns out that a similar structure exists when excluding a star as an $\mathcal{F}$-rich minor.

Let G be a graph, let $\mathcal{F}$ be a family of connected subgraphs of G, let $\mathcal{D} = \big(T, (W_x \mid x \in V(T))\big)$ be a tree decomposition of G, and let H be a subgraph of G. We denote by $\mathcal{F}|_H$ the family $\{F \in \mathcal{F} \mid F \subseteq H\}$. Moreover, we denote by $\mathcal{D}|_H$ the tree decomposition $\big(T, (W_x \cap V(H) \mid x \in V(T))\big)$ of H.

Let G be a graph and let $S \subseteq V(G)$. A *path partition of* (G, S) if an ordered partition $(P_0, \dots, P_m)$ of S such that

(pp1) for every $i, j \in [m]$, if there is an edge $uv \in E(G)$ with $u \in P_i$ and $v \in P_j$, then $|i - j| \leqslant 1$, and

(pp2) for every component C of $G - S$, there exists $i, j \in [m]$ with $|i - j| \leqslant 1$ such that $N_G(V(C)) \subseteq P_i \cup P_j$.

See Figure 3.7. The *width* of $(P_1, \dots, P_m)$ is $\max_{i \in [m]} |P_i|$. The *path partition width of* (G, S), denoted by $\mathrm{pp}(G, S)$, is the minimum width of a path partition of (G, S).

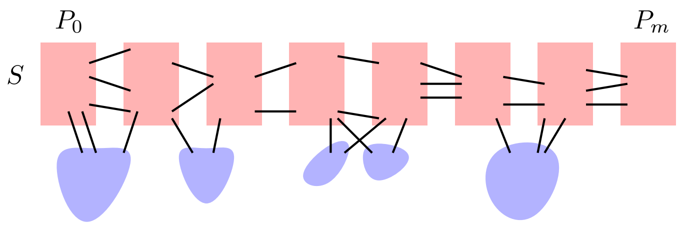


Figure 3.7: A path partition $(P_0, \dots, P_m)$ of (G, S). In red is depicted the set S and in blue the components of $G - S$.

Theorem 3.13. *Let ℓ, t be positive integers, let G be a graph with $\operatorname{tw}(G) < t$, and let $\mathcal{F}$ be a family of connected subgraphs of G. If G has no $\mathcal{F}$-rich model of $K_{1,\ell}$, then there exists $S \subseteq V(G)$ such that*

(A) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and*

(B) *$\operatorname{pp}(G, S) \leqslant t\ell$.*

This statement was inspired by a result of Dujmović, Hickingbotham, Joret, Micek, Morin, and Wood [DHJ+23], that we will prove later in this section. Theorem 3.13 is implied by the following statement that we prove by induction.

Lemma 3.14. *Let ℓ be a positive integer, let G be a connected graph, let $\mathcal{D}$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $K_{1,\ell}$. For every set $R \subseteq V(G)$ such that $G[R]$ is connected, there exists $S \subseteq V(G)$ and a path partition $(P_1, \dots, P_m)$ of (G, S) such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,*

(b) *$P_1 = R$, and*

(c) *P_i is contained in a union of at most ℓ bags of $\mathcal{D}$, for every $i \in \{2, \dots, m\}$.*

Proof. Let $R \subseteq V(G)$ such that $G[R]$ is connected. We proceed by induction on $|V(G - R)|$. If $\mathcal{F}|_{G-R} = \emptyset$, then let $m = 1$ and $S = P_1 = R$. Now suppose $\mathcal{F}|_{G-R} \neq \emptyset$ and suppose that the result holds for smaller values of $|V(G - R)|$. In particular $V(G - R) \neq \emptyset$.

Let $\mathcal{F}'$ be the family of all the connected subgraphs H of $G - R$ such that $N_G(R) \cap V(H) \neq \emptyset$ and there exists $F \in \mathcal{F}$ such that $F \subseteq H$. We claim that there are no $\ell + 1$ disjoint members of $\mathcal{F}'$.

Suppose for contradiction that there exists $H_1, \dots, H_{\ell+1} \in \mathcal{F}'$ pairwise disjoint. Then $(V(H_1), \dots, V(H_\ell), R \cup V_{\ell+1})$ is an $\mathcal{F}$-rich model of $K_{1,\ell}$ with the branch set of the center being $R \cup V(H_{\ell+1})$. This proves that there are no $\ell + 1$ pairwise disjoint members of $\mathcal{F}'$.

Hence, by Lemma 1.17, there exists $Z \subseteq V(G)$ contained in a union of at most ℓ bags of $\mathcal{D}$ intersecting every member of $\mathcal{F}'$. Take such a set Z which is inclusion-wise minimal. Let U be the union of the vertex sets of all the connected components C of $G - (R \cup Z)$ containing at least one member of $\mathcal{F}$, that is such that $\mathcal{F}|_C \neq \emptyset$. Let $R' = V(G) \setminus U$. We claim that $G[R']$ is connected. Since G is connected, every component of $G[R'] - (R \cup Z)$ has a neighbor in $R \cup Z$. Therefore, as $G[R]$ is connected, it is enough to show that for every $u \in Z$, there is a path from u to R in

$G[R'] = G - U$. Let $u \in Z$. Since $Z \setminus \{u\}$ does not intersect every member of $\mathcal{F}'$, there exists $H \in \mathcal{F}'$ such that $H \cap Z = \{u\}$. But by definition of $\mathcal{F}'$, H contains a path from $N_G(R)$ to u. By construction, this path must be disjoint from U. Hence there is a path from u to R disjoint from U in G. We deduce that $G[R']$ is connected.

Moreover, since $\mathcal{F}|_{G-R} \neq \emptyset$ and G is connected, $\mathcal{F}' \neq \emptyset$, which implies $Z \neq \emptyset$ and so $|V(G - R')| < |V(G - R)|$. Hence by the induction hypothesis applied to R', there exists $S' \subseteq V(G)$ and a path partition $(P'_1, \dots, P'_{m'})$ of (G, S') such that

(a') $V(F) \cap S' \neq \emptyset$ for every $F \in \mathcal{F}$,

(b') $P'_1 = R'$, and

(c') P'_i is contained in a union of at most ℓ bags of $\mathcal{D}$, for every $i \in \{2, \dots, m'\}$.

See Figure 3.8.

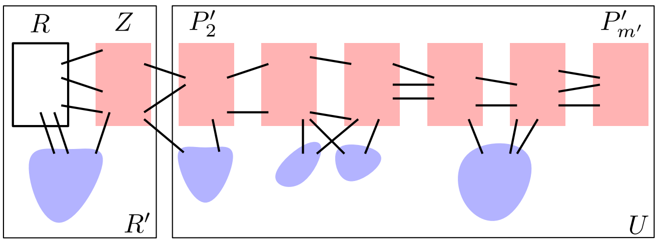


Figure 3.8: Illustration for the proof of Lemma 3.14. In blue are depicted the components of $G - S$.

We can now define S and its partition $(P_1, \dots, P_m)$. Let

$$m = m' + 1, \quad S = (S' \setminus R') \cup R \cup Z$$
$$P_1 = R, \quad P_2 = Z$$
$$P_i = P'_{i-1} \quad \text{for every } i \in \{3, \dots, m' + 1\}.$$

We claim that $(P_1, \dots, P_m)$ satisfies is a path partition of (G, S) satisfying (a) to (c).

First we show that $(P_1, \dots, P_m)$ is a path partition of (G, S). We start by proving (pp1). Let $i, j \in [m]$ and let $u \in P_i, v \in P_j$ such that $uv \in E(G)$. Without loss of generality, $i \leqslant j$. If $i = 1$, then $v \in R \cup N_G(R)$ since $P_1 = R$. If $j > 2$, then $v \in P'_2 \cup \dots \cup P'_{m'} = U$. But by the definition of U, the component C of u in $G - (R \cup Z)$ contains a member of $\mathcal{F}$, which implies that $C \in \mathcal{F}'$. Since C is disjoint from Z, this contradicts the fact that Z intersects every member of $\mathcal{F}'$. This proves $j \leqslant 2$. Now suppose that $i \geqslant 2$. In particular $u \in P_i \subseteq P'_{i-1}$. Then since $(P'_1, \dots, P'_{m'})$ is a path partition of (G, S'), this implies that $v \in P'_{i-1} \cup P'_i$, and so $v \in P_i \cup P_{i+1}$. This proves that $j \leqslant i + 1$. Therefore, (pp1) holds.

To prove (pp2), consider a component C of $G - S$. If C intersects R', then by the definition of U, $N_G(V(C)) \subseteq R \cup Z = P_1 \cup P_2$. Now suppose that C is disjoint from R'. Then, since $(P'_1, \dots, P'_{m'})$ is a path partition of (G, S'), there exist $i, j \in [m']$ with $i \leqslant j \leqslant i + 1$ such that $N_G(V(C)) \subseteq P'_i \cup P'_j$. If $i \geqslant 2$, then $P'_i \cup P'_j = P_{i+1} \cup P_{j+1}$ and so $N_G(V(C)) \subseteq P_{i+1} \cup P_{j+1}$. Now suppose $i = 1$. Then since $N_G(U) \subseteq Z$, we conclude that $N_G(V(C)) \subseteq Z \cup P'_j = P_2 \cup P_{j+1}$. This proves (pp2) and so $(P_1, \dots, P_m)$ is a path partition of (G, S).

To prove (a), consider $F \in \mathcal{F}$. By (a'), there exists $i \in [m']$ such that $P_i \cap V(F) \neq \emptyset$. If $i \geqslant 2$, then $V(F) \cap P_{i+1} \neq \emptyset$ and so $V(F) \cap S \neq \emptyset$. Now suppose that $V(F) \cap P'_1 \neq \emptyset$. If $V(F) \cap R \neq \emptyset$, then $V(F) \cap S \neq \emptyset$. Otherwise, let C be the component of $G - R$ containing F. Then by the definition of $\mathcal{F}'$, $C \in \mathcal{F}'$. As a consequence, either $V(F) \cap Z \neq \emptyset$ and so $V(F) \cap S \neq \emptyset$, or $V(F) \subseteq U$. But then $V(F)$ is disjoint from $P'_1 = R'$, a contradiction. This proves that $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so (a) holds.

By construction, $P_1 = R$ and so (b) holds. Moreover, since Z is contained in a union of at most ℓ bags of $\mathcal{D}$, as well as P_i for every $i \in \{3, \dots, m\}$ by (c'), (c) holds. This concludes the proof of the lemma. □

Proof of Theorem 3.13. We will apply Lemma 3.14 to every component of G for R being an arbitrary singleton and $\mathcal{D}$ a tree decomposition of width less than t. Let $C_1, \dots, C_c$ be the components of G. Let $j \in [c]$, and let R^j be an arbitrary singleton in $V(C_j)$. By Lemma 3.14, there exists $S_j \subseteq V(G)$ and a path partition $(P^j_1, \dots, P^j_{m^j})$ of (C_j, S_j) such that

3.14.(a) $V(F) \cap S_j \neq \emptyset$ for every $F \in \mathcal{F}|_{C_j}$,

3.14.(b) $P^j_1 = R^j$, and

3.14.(c) $|P^j_i| \leqslant t\ell$ for every $i \in \{2, \dots, m\}$.

Then let

$$S = \bigcup_{j \in [c]} S_j \quad \text{and} \quad (P_1, \dots, P_m) = (P^1_1, \dots, P^1_{m^1}, P^2_1, \dots, P^c_{m^c}).$$

The sequence $(P_1, \dots, P_m)$ is a path partition of (G, S), and 3.14.(a) implies that $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$. This proves the theorem. □

This proof is a first glimpse of the usefulness of $\mathcal{F}$-rich models. By applying iteratively the result for graphs with no $\mathcal{F}$-rich model of $\overline{K_\ell}$, we obtain a result for graphs with no $\mathcal{F}$-rich models of $K_1 \oplus \overline{K_\ell} = K_{1,\ell}$. Below, we will see that we can similarly apply this result to deduce a result for graphs with no $\mathcal{F}$-rich model of a given forest.

Excluding a forest. In [DHJ+23], Dujmović, Hickingbotham, Joret, Micek, Morin, and Wood proved the following inspiring result.

Theorem 3.15 (Dujmović, Hickingbotham, Joret, Micek, Morin, and Wood [DHJ+23])**.** *Let h be an integer with $h \geqslant 2$, and let X be a tree of vertex-height h. There exists a positive integer c such that for every graph G, if X is not a minor of G, then there exists a graph H such that*

(A) *$G \subseteq H \boxtimes K_c$, and*

(B) $\mathrm{pw}(H) < 2(h+1)$.

We prove this theorem within the framework of rich models. While the presentation is very different from the original paper [DHJ+23], the proof is fundamentally the same. Let h, d be positive integers. Let $F_{h,d}$ be the complete d-ary tree of vertex-height h. See Figure 3.9. In particular, $F_{2,d} = K_{1,d}$. We root $F_{h,d}$ naturally. Note that for every tree X of vertex-height h, there exists a positive integer d such that $X \subseteq F_{h,d}$. Hence, it is enough to prove Theorem 3.15 for X of the form $F_{h,d}$.

We will use the following property of the family $(F_{h,d})_{h,d \in \mathbb{N}_{>0}}$.

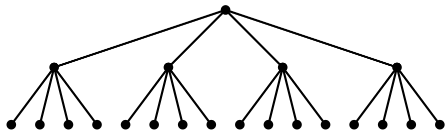


Figure 3.9: The tree $F_{3,4}$.

Lemma 3.16. *Let h, d be positive integers. Let G be a connected graph. If there is a model $(B_x \mid x \in V(F_{h,d+1}))$ of $F_{h,d+1}$ in G, then for every $u \in V(G)$, there is a model $(B'_x \mid x \in V(F_{h,d}))$ of $F_{h,d}$ in G such that*

(a) *$u \in B'_s$, where s is the root of $F_{h,d}$, and*

(b) *for every $x \in V(F_{h,d})$, $B_y \subseteq B'_x$ for some $y \in V(F_{h,d+1})$.*

Proof. Suppose that there is a model $(B_x \mid x \in V(F_{h,d+1}))$ of $F_{h,d+1}$ in G. Since G is connected, we can assume that $\bigcup_{x \in V(F_{h,d+1})} B_x = V(G)$. Let s_0 be the root of $F_{h,d+1}$. There is a subtree T' of $F_{h,d+1}$ rooted in a child of s_0 such that $u \in \bigcup_{x \in V(T') \cup \{s_0\}} B_x$. Define $B'_s = \bigcup_{x \in V(T') \cup \{s_0\}} B_x$ and $B'_x = B_x$ for every $x \in V(F_{h,d+1}) \setminus (\{s_0\} \cup V(T'))$. The collection $(B'_x \mid x \in V(F_{h,d}))$ is a model of $F_{h,d}$ in G satisfying (a) and (b). □

Let G be a graph and let $\mathcal{P}$ be a partition of $V(G)$. We denote by $G/\mathcal{P}$ the graph with vertex set $\mathcal{P}$ and edges all the pairs $P, P' \in \mathcal{P}$ such that there is an edge $uv \in E(G)$ with $u \in P$ and $v \in P'$. Note that for every graph H, $G \subseteq H \boxtimes K_c$ if and only if there is a partition $\mathcal{P}$ of $V(G)$ such that

(i) $G/\mathcal{P} \subseteq H$, and

(ii) every part of $\mathcal{P}$ has size at most c.

Before proving Theorem 3.15, we define the following variant of product structure "focused" on a subset S of vertices. A *path decomposition of* $(G, \mathcal{P})$ is a sequence $(W_1, \dots, W_m)$ where $W_i \subseteq \mathcal{P}$ for every $i \in [m]$ such that

(pd1) for every $u \in S$, $\{i \in [m] \mid u \in \bigcup W_i\}$ is a nonempty interval of integers,

(pd2) for every edge uv of $G[S]$, there exists $i \in [m]$ such that $u, v \in \bigcup W_i$, and

(pd3) for every component C of $G - S$, there exists $i \in [m]$ such that $N_G(V(C)) \subseteq \bigcup W_i$.

The *width* of this path decomposition is then $\max_{i \in [m]} |W_i| - 1$, and we define the *pathwidth of* $(G, \mathcal{P})$, denoted by $\mathrm{pw}(G, \mathcal{P})$, as the minimum width of a path decomposition of $(G, \mathcal{P})$. We prove by induction on h the following lemma, which will immediately implies Theorem 3.15.

Lemma 3.17. *Let h, d be integers with $h, d \geqslant 2$. There is an integer $c_{3.17}(h, d)$ such that for every positive integer t, for every graph G with $\mathrm{tw}(G) < t$, for every family $\mathcal{F}$ of connected subgraphs of G, if there is no $\mathcal{F}$-rich model of $F_{h,d}$ in G, then there exists $s \subseteq V(G)$ and a partition $\mathcal{P}$ of S such that*

(A) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,*

(B) *$|P| \leqslant c_{3.17}(h,d)t$ for every $P \in \mathcal{P}$, and*

(C) $\mathrm{pw}(G, \mathcal{P}) < 2(h+1)$.

Proof. We proceed by induction on h. If $h = 2$, then $F_{h,d} = K_{1,d}$ and so the result holds by Lemma 3.14 for $c_{3.17}(1,d) = d$. Now suppose $h > 2$ and that the result holds for $h-1$. Let

$$c_{3.17}(h,d) = \max\{d, c_{3.17}(h-1, d+1)\}.$$

Let $\mathcal{F}'$ be the family of all the connected subgraphs H of G such that H has an $\mathcal{F}|_H$-rich model of $F_{h-1,d+1}$. We claim that there are no $\mathcal{F}'$-rich model of $K_{1,d}$ in G.

Suppose for a contradiction that there is an $\mathcal{F}'$-rich model $\left(B_x \mid x \in V(K_{1,d})\right)$ of $K_{1,d}$ in G. Figure 3.10 illustrates how to deduce an $\mathcal{F}$-rich model of $F_{h,d}$. We denote by c the center of the star $K_{1,d}$ and by $x_1, \dots, x_d$ the leaves of $K_{1,d}$. We denote by r the root of $F_{h-1,d+1}$. Let $i \in [d]$. Since $\left(B_x \mid x \in V(K_{1,d})\right)$ is $\mathcal{F}'$-rich, B_{x_i} contains a neighbor u_i of B_c, and $G[B_{x_i}]$ has an $\mathcal{F}|_H$-rich model $\left(B^i_x \mid x \in V(F_{h-1,d+1})\right)$ of $F_{h-1,d+1}$. By Lemma 3.16, there is an $\mathcal{F}$-rich model $\mathcal{M}_i$ of $F_{h-1,d}$ in $G[B_{x_i}]$ whose branch set of the root contains a neighbor of B_c. It follows that the union of the models $\mathcal{M}_1, \dots, \mathcal{M}_d$, together with B_c for the branch set of the root, yields an $\mathcal{F}$-rich model of $F_{h,d}$ in G, a contradiction. This proves that there is no $\mathcal{F}'$-rich model of $K_{1,d}$ in G.

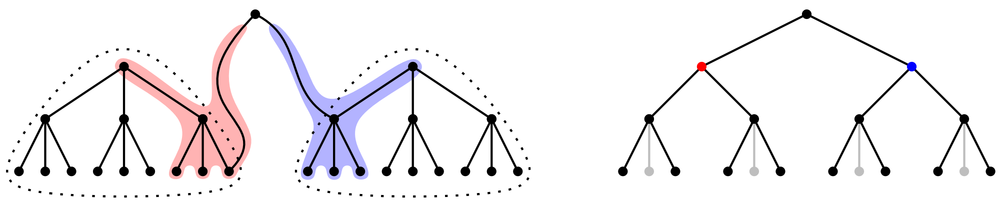


Figure 3.10: In the proof of Lemma 3.17, an $\mathcal{F}'$-rich model of $K_{1,d}$ yields an $\mathcal{F}$-rich model of $F_{h,d}$. This model is constructed by contracting the colored sets, and removing the gray vertices and edges.

Hence, by Lemma 3.14, there exists $S_0 \subseteq V(G)$ such that

3.14.(A) $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}'$, and

3.14.(B) $\mathrm{pp}(G, S_0) \leqslant dt$.

Let $(P_1, \dots, P_m)$ be a path partition of (G, S_0) of width at most dt. By convention, we set $P_0 = P_{m+1} = \emptyset$. By definition of path partition, for every component C of $G - S_0$, there exists $i_C \in [m]$ such that $N_G(V(C)) \subseteq P_{i_C} \cup P_{i_C+1}$. For every $i \in [m]$, let $U_i = \bigcup_{C \in \mathcal{C}, i_C = i} V(C)$ where $\mathcal{C}$ is the family of all the connected components of $G - S_0$. By convention, we set $U_0 = \emptyset$.

Let $i \in [m]$. By the definition of $\mathcal{F}'$, the graph $G[U_i]$ has no $\mathcal{F}$-rich model of $F_{h-1,d+1}$. Hence, by the induction hypothesis, there exists $S^i \subseteq U_i$ and a partition $\mathcal{P}^i$ of S^i such that

(A') $V(F) \cap S^i \neq \emptyset$ for every $F \in \mathcal{F}|_{G[U_i]}$,

(B') $|P| \leqslant c_{3.17}(h-1, d+1)t$ for every $P \in \mathcal{P}^i$, and

(C') $\mathrm{pw}(G[U_i], \mathcal{P}^i) < 2h$.

Let $(W^i_1, \dots, W^i_{m^i})$ be a path decomposition of $(G[U_i], \mathcal{P}^i)$ of width less than $2h$. By possibly adding an empty bag, we assume that $m^i > 0$. For every $j \in [m^i]$, let

$$W_{i,j} = \{P_i, P_{i+1}\} \cup W^i_j.$$

See Figure 3.11. We take

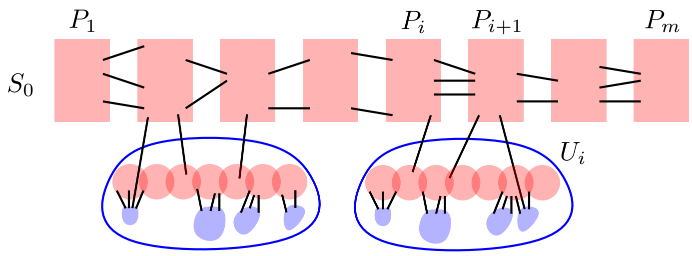


Figure 3.11: Illustration for the proof of Lemma 3.17. The components of $G - S$ are depicted in blue.

$$S = S_0 \cup \bigcup_{i \in [m]} S^i, \quad \text{and} \quad \mathcal{P} = \{P_1, \dots, P_m\} \cup \bigcup_{i \in [m]} \mathcal{P}^i.$$

Let $I = \{(i,j) \mid i \in [m], j \in [m^i]\}$. We order I lexicographically. Let $\mathcal{D}$ be the sequence $(W_{i,j})_{(i,j) \in I}$. We claim that $\mathcal{D}$ is a path decomposition of $(G, \mathcal{P})$, and that $S, \mathcal{P}, \mathcal{D}$ satisfies (A), (B), and (C).

First, we show that $\mathcal{D}$ is a path decomposition of $(G, \mathcal{P})$. Let $u \in S$. If $u \in S_0$, then

$$\{(i,j) \in I \mid u \in \bigcup W_{i,j}\} = \{(i,j) \mid i \in [m], u \in P_i \cup P_{i+1}, j \in [m^i]\},$$

which is a nonempty interval of I. If $u \in S^i$ for some $i \in [m]$, then

$$\{(i,j) \in I \mid u \in \bigcup W_{i,j}\} = \{i\} \times \{j \in [m^i] \mid u \in \bigcup W^i_j\},$$

which is a nonempty interval of integers because as $(W^i_1, \dots, W^i_{m^i})$ is a path decomposition of $(G[U_i], \mathcal{P}^i)$, the set of indices $\{j \in [m^i] \mid u \in \bigcup W^i_j\}$ is a nonempty interval of integers. This proves (pd1).

Let uv be an edge of $G[S]$. First suppose that $u \in S_0$. Then, there exists $i \in [m]$ such that $u \in P_i$. Since $(P_1, \dots, P_m)$ is a path partition of (G, S_0), and by the definition of U_{i-1}, U_i,

$$v \in P_{i-1} \cup U_{i-1} \cup P_i \cup U_i \cup P_{i+1}.$$

If $v \in P_{i-1} \cup P_i$, then $u, v \in \bigcup W_{i-1,1}$. If $v \in P_{i+1}$, then $u, v \in \bigcup W_{i,1}$. If $v \in U_{i-1}$, then $i \geqslant 1$, and since $(W^{i-1}_1, \dots, W^{i-1}_{m^{i-1}})$ is a path decomposition of $(G[U_{i-1}], \mathcal{P}_{i-1})$, there exists $j \in [m^{i-1}]$ such that $v \in \bigcup W^{i-1}_j$, and so $u, v \in \bigcup W_{i-1,j}$. Finally, if $v \in U_i$, then since $(W^i_1, \dots, W^i_{m^i})$ is a path decomposition of $(G[U_i], \mathcal{P}_i)$, there exists $j \in [m^i]$ such that $v \in \bigcup W^i_j$, and so $u, v \in \bigcup W_{i,j}$. This concludes the case $u \in S_0$. The case $v \in S_0$ is symmetric. Now suppose that $u, v \notin S_0$. Then there exists $i \in [m]$ such that $u, v \in S^i$. Since $(W^i_1, \dots, W^i_{m^i})$ is a path decomposition of $(G[U_i], \mathcal{P}^i)$, there exists $j \in [m^i]$ such that $u, v \in \bigcup W^i_j \subseteq \bigcup W_{i,j}$. This proves (pd2).

Let C be a component of $G - S$. Let C' be the component of $G - S_0$ containing C. There exists $i \in [m]$ such that $V(C') \subseteq U_i$, and so C is a component of $G[U_i] - S^i$. Since $(W^i_1, \ldots, W^i_{m^i})$ is a path decomposition of $(G[U_i], \mathcal{P}^i)$, there exists $j \in [m^i]$ such that $N_{C'}(V(C)) \subseteq \bigcup W^i_j$. Moreover, $N_G(V(C)) \subseteq N_{C'}(V(C)) \cup N_G(V(C))$, and since $N_G(V(C')) \subseteq P_i \cup P_{i+1}$ by definition of U_i, we conclude that

$$N_G(V(C)) \subseteq (\bigcup W^i_j) \cup P_i \cup P_{i+1} = \bigcup W_{i,j}.$$

This proves (pd3), and so $\mathcal{D}$ is a path decomposition of $(G, \mathcal{P})$.

We now prove (A). For every $F \in \mathcal{F}$, either $V(F) \cap S_0 \neq \emptyset$, or there exists $i \in [m]$ such that $F \subseteq G[U_i]$, which implies $V(F) \cap S^i \neq \emptyset$ by (A'). In both cases, we deduce that $V(F) \cap S \neq \emptyset$. Therefore, (A) holds.

To show (B), observe that for every $P \in \mathcal{P}$, either $P \in \{P_1, \ldots, P_m\}$ and so $|P| \leqslant dt$ by 3.14.(B), or $P \in \mathcal{P}^i$ for some $i \in [m]$ and so $|P| \leqslant c_{3.17}(h-1, d+1)t$ by (B'). In both cases, $|P| \leqslant c_{3.17}(h, d)t$ and therefore (B) holds.

Finally, for every $(i, j) \in I$,

$$|W_{i,j}| = 2 + |W^i_j| \leqslant 2 + 2h = 2(h+1)$$

by (C'), and so (C) holds. This concludes the proof of the theorem. □

Proof of Theorem 3.15. Let X be a graph of vertex-height at most h. There exists a positive integer d such that $X \subseteq F_{h,d}$. Let

$$c = c_{3.17}(h, d)(|V(X)| - 1).$$

Let G be an X-minor-free graph. By Theorem 1.11, $\mathrm{tw}(G) \leqslant \mathrm{pw}(G) < |V(X)| - 1$. Let $\mathcal{F}$ be the family of all the connected subgraphs of G. In particular, there is a no $\mathcal{F}$-rich model of $F_{h,d}$ in G. By Lemma 3.17, this implies that there exists $S \subseteq V(G)$ and a partition $\mathcal{P}$ of S such that

(A) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,

(B) $|P| \leqslant c_{3.17}(h, d)(|V(X)| - 1)$ for every $P \in \mathcal{P}$, and

(C) $\mathrm{pw}(G, \mathcal{P}) < 2h$.

By (A) and the definition of $\mathcal{F}$, we have $S = V(G)$. Let $H = G/\mathcal{P}$. Then (B) implies $G \subseteq H \boxtimes K_c$, and (C) gives $\mathrm{pw}(H) < 2h$. This proves the theorem. □

Interestingly, Lemma 3.17 has applications beyond product structure. Indeed, it implies that for any tree X, there exists a positive integer c such that for every graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if there is no $\mathcal{F}$-rich model of X in G, then there is hitting set S of $\mathcal{F}$ with $\mathrm{pw}(G, S) < c(\mathrm{tw}(G) + 1)$. Choosing carefully $\mathcal{F}$, we can obtain a few non trivial results. Consider a graph Y that consists of a tree X, with triangles added on every leaf. See Figure 3.12. Now, if G does not contain Y as a minor, then it has no $\mathcal{F}$-rich model of X for $\mathcal{F}$ being the family of all the connected subgraphs of G containing K_3 as a minor. By Lemma 3.17, this implies that

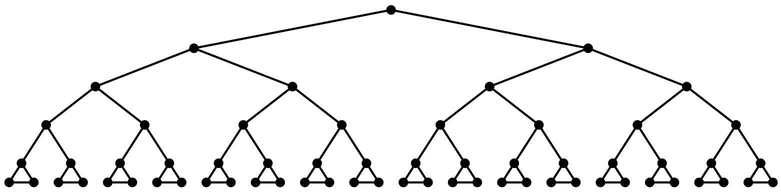


Figure 3.12: A graph obtained from a tree by adding a triangle on every leaf.

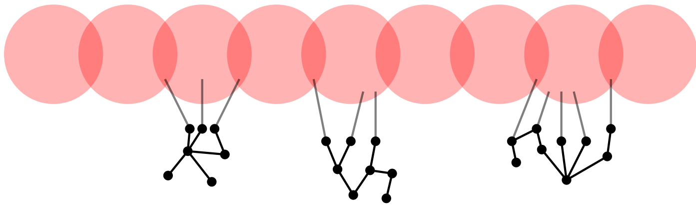


Figure 3.13: A feedback vertex set S (in red), and a path decomposition of (G, S).

there is a hitting set S for $\mathcal{F}$ with $\mathrm{pw}(G, S) < c(\mathrm{tw}(G) + 1)$. Since Y is planar, the treewidth of G is bounded by a function of Y by the Grid-Minor Theorem (Theorem 1.10). Therefore, G has a set S of vertices intersecting every cycle of G, i.e. a *feedback vertex set* of G, with $\mathrm{pw}(G, S)$ bounded. See Figure 3.13. An simple observation is that there are such graphs Y for which $\min_{S \subseteq V(Y) \text{ feedback vertex set of } Y} \mathrm{pw}(G, S)$ is arbitrarily large. Therefore, this is a characterization in term of excluded minors of graph classes for which the minor-monotone graph parameter

$$G \mapsto \min_{S \subseteq V(G) \text{ feedback vertex set of } G} \mathrm{pw}(G, S)$$

is bounded.

CHAPTER 4

Layered parameters

This chapter contains joint work with Jędrzej Hodor, Hoang La, and Piotr Micek, and is based on [HLMR24b].

Let G be a graph. Recall that a *layering* of G is a sequence $(L_i \mid i \geqslant 0)$ of disjoint subsets of $V(G)$ whose union is $V(G)$ and such that for every $uv \in E(G)$ there is a non-negative integer i such that $u, v \in L_i \cup L_{i+1}$.

Let $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ be a tree decomposition of G and let $\mathcal{L} = (L_i \mid i \in \mathbb{N})$. The *width* of $(\mathcal{D}, \mathcal{L})$ is $\max\{|W_x \cap L_i| \mid x \in V(T),\ i \geqslant 0\}$. The *layered treewidth* of G, denoted by $\mathrm{ltw}(G)$, is the minimum width of a pair $(\mathcal{D}, \mathcal{L})$, where $\mathcal{D}$ is a tree decomposition of G and $\mathcal{L}$ is a layering of G. The *layered pathwidth* of G, denoted by $\mathrm{lpw}(G)$ is the minimum width of a pair $(\mathcal{D}, \mathcal{L})$, where $\mathcal{D}$ is a path decomposition of G and $\mathcal{L}$ is a layering of G.

A graph is an *apex graph* if it can be made planar by the removal of at most one vertex, and a graph is an *apex-forest* if it can be made acyclic by the removal of at most one vertex. It turns out that forbidding apex-type graphs as minors interplays with the layered versions of treewidth, pathwidth, and treedepth. Dujmović, Morin, and Wood [DMW17] proved that a minor-closed class of graphs excludes an apex graph if and only if it has bounded layered treewidth. Similarly, Dujmović, Eppstein, Joret, Morin, and Wood in [DEJ+20a], proved that a minor-closed class of graphs excludes an apex-forest if and only if it has bounded layered pathwidth.

In this chapter, we give a short and simple proof of the latter statement with an explicit and much better bound on layered pathwidth.

Theorem 1.23. *Let X be an apex-forest with at least two vertices. For every graph G, if G is X-minor-free, then*

$$\mathrm{lpw}(G) \leqslant 2|V(X)| - 3.$$

We propose a natural counterpart of the definitions above for treedepth. Let F be an elimination forest of G, and let $\mathcal{L} = (L_i \mid i \geqslant 0)$ be a layering of G. The *width* of $(F, \mathcal{L})$ is $\max\{|R \cap L_i| \mid R \text{ is a root-to-leaf path in } F, i \geqslant 0\}$. The *layered treedepth* of G, denoted by

$\operatorname{ltd}(G)$, is the minimum width of a pair $(F, \mathcal{L})$, where F is an elimination forest of G and $\mathcal{L}$ is a layering of G.

A graph is a *fan* or (an *apex-path*) if it becomes a path by the removal of at most one vertex. Since a layering of a fan has at most 3 nonempty layers, and since paths have large treedepth (see the introduction), we deduce that fans have unbounded layered treedepth. We show that reciprocally, in the context of minor-closed classes of graphs, if a class of graphs excludes a fan as a minor, then it has bounded layered treedepth.

Theorem 1.24. *For every fan X with at least three vertices, and for every graph G, if G is X-minor-free, then*

$$\operatorname{ltd}(G) \leqslant \binom{|V(X)|-1}{2}.$$

4.1 Preliminaries

A *separation* of G is a pair (A, B) of subgraphs of G such that $A \cup B = G$, $E(A \cap B) = \emptyset$. The *order* of (A, B) is $|V(A) \cap V(B)|$. We need the following well-known version of Menger's Theorem.

Theorem 4.1 (Menger's Theorem for separations)**.** *Let G be a graph and $X, Y \subseteq V(G)$. There exists a separation (A, B) of G such that $X \subseteq V(A)$, $Y \subseteq V(B)$, and there exists $|V(A) \cap V(B)|$ pairwise disjoint (X, Y)-paths.*

Let G be a graph and let $S \subseteq V(G)$. If H is a plane graph, we say that a model $(B_x \mid x \in V(H))$ of H in G is *S-outer-rooted* if $B_x \cap S \neq \emptyset$ for each vertex x in the outer face of H.

4.2 Excluding a rooted path and layered treedepth

Recall that if a graph G has no model of P_ℓ, then $\operatorname{td}(G) < \ell$. We prove an analogous result within the setting of S-rooted models of paths.

Theorem 4.2. *For every positive integer ℓ, for every graph G, and for every $S \subseteq V(G)$, if G has no S-rooted model of P_ℓ, then $\overline{\operatorname{td}}(G, S) \leqslant \binom{\ell}{2}$.*

As already mentioned, Theorem 4.2 is the main ingredient of the proof of Theorem 1.24. Actually, the intuition standing behind this is very simple. For a vertex u in a graph G, we set $S = N(u)$. Now, if $G - u$ has a S-rooted model of a path P_ℓ, then G has a model of P_ℓ with a universal vertex added, and so G has a model of every fan on $\ell + 1$ vertices.

In this section, we prove Theorem 4.2 and Theorem 1.24. In Chapter 3, we stated the definition of treedepth via elimination trees. Treedepth can be also equivalently defined inductively. Namely, treedepth of a graph is the maximum of treedepth of its components, treedepth of the one-vertex graph is 1, and when a graph G has more than one vertex and is connected, treedepth is the minimum over all vertices $v \in V(G)$ of $\operatorname{td}(G - v) + 1$. The new version of treedepth that we proposed also admits an inductive definition, which we state in terms of properties (t1) to (t3). Let G be a graph, and $S \subseteq V(G)$.

(t1) If $S = \emptyset$, then $\overline{\mathrm{td}}(G, S) = 0$.

(t2) If $V(G) \neq \emptyset$, then $\overline{\mathrm{td}}(G, S) = \max_C \overline{\mathrm{td}}(C, S \cap V(C))$, where C goes over all components of G.

(t3) If G is connected and $S \neq \emptyset$, then $\overline{\mathrm{td}}(G, S) = 1 + \min_{u \in V(G)} \overline{\mathrm{td}}(G - u, S \setminus \{u\})$.

It immediately follows that treedepth is monotone in the following sense.

(t4) If H is a subgraph of G, then $\overline{\mathrm{td}}(H, S \cap V(H)) \leqslant \overline{\mathrm{td}}(G, S)$.

A *depth-first-search tree*, *DFS tree* for short, of G is a rooted spanning tree T of G such that T is an elimination forest of G. We proceed with the proof of Theorem 4.2, the key inductive step is encapsulated in the following lemma.

Lemma 4.3. *Let G be a connected graph, let $S \subseteq V(G)$, and let T be a DFS tree of G. For every positive integer ℓ, if for each root-to-leaf path P in T, there are no ℓ pairwise disjoint $(V(P), S)$-paths in G, then*

$$\overline{\mathrm{td}}(G, S) \leqslant \binom{\ell}{2}.$$

Proof. We proceed by induction on ℓ. If $\ell = 1$, then $S = \emptyset$, and so, $\mathrm{td}(G, S) = 0$ by (t1). Now, assume that $\ell \geqslant 2$.

For every $u \in V(G)$, let T_u be the subtree of T rooted in u, and let $G_u = G[V(T_u)]$. Let $s_0 \in V(G)$ be the vertex with maximum depth in T such that $S \subseteq V(T_{s_0})$. Let R be the path from the root to s_0 in T. By assumption, there are no ℓ pairwise disjoint $(V(R), S)$-paths. Hence by Menger's Theorem, there is a separation (A, B) of G of order at most $\ell - 1$ such that $V(R) \subseteq V(A)$ and $S \subseteq V(B)$. In particular, every $(V(R), S)$-path intersects $X = V(A) \cap V(B)$.

Consider a component C of $G - X$. If C has no vertex in S, then $\mathrm{td}(C, S \cap V(C)) = 0$. Therefore, we assume the opposite, namely, $V(C) \subseteq V(T_{s_0}) \setminus \{s_0\}$. It follows that there is a child v of s_0 with $V(C) \subseteq V(T_v)$. The next goal is to apply induction to G_v – this step is illustrated in Figure 4.1. To this end, we claim that for every root-to-leaf path P' in T_v there are no $\ell - 1$ pairwise disjoint $(V(P'), S)$-paths in G_v. Suppose to the contrary that there is such a root-to-leaf path P'. Let P be the path connecting the root of T and the unique leaf in P'. By the maximality of s_0, we have $S \not\subseteq V(T_v)$. Hence, there is $w \in S$ such that $w \notin V(T_v)$. Let Q be the shortest path from s_0 to w in T. Note that Q is a $(V(R), S)$-path, and Q disjoint from $V(T_v)$. Therefore, the $\ell - 1$ pairwise disjoint $(V(P'), S)$-paths in G_v and Q form a collection of ℓ pairwise disjoint $(V(P), S)$-paths in G, which is a contradiction. By inductive hypothesis applied to G_v and T_v we obtain $\overline{\mathrm{td}}(G_v, S \cap V(G_v)) \leqslant \binom{\ell-1}{2}$. By (t4), this yields $\overline{\mathrm{td}}(C, S \cap V(C)) \leqslant \binom{\ell-1}{2}$ for every component of $G - X$.

By (t2), we deduce $\overline{\mathrm{td}}(G - X, S - X) \leqslant \binom{\ell-1}{2}$. Finally, by (t3),

$$\overline{\mathrm{td}}(G, S) \leqslant |X| + \overline{\mathrm{td}}(G - X, S \setminus X) \leqslant (\ell - 1) + \binom{\ell - 1}{2} = \binom{\ell}{2}. \qquad \square$$

Proof of Theorem 4.2. Let ℓ be a positive integer, let G be a graph, and let $S \subseteq V(G)$. We can assume that G is connected due to (t2). Assume that G has no S-rooted model of P_ℓ, and suppose to the contrary that $\mathrm{td}(G, S) > \binom{\ell}{2}$. Then, by Lemma 4.3 applied with an arbitrary DFS-tree of G, there is a path P in G and ℓ pairwise disjoint $(V(P), S)$-paths in G. These paths, together with P, give an S-rooted model of P_ℓ in G. This is a contradiction, which ends the proof. $\square$

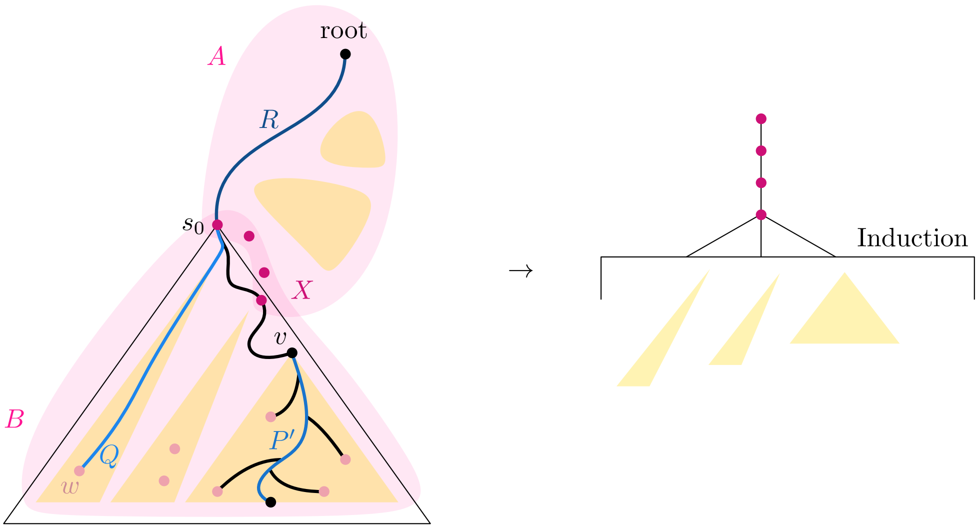


Figure 4.1: An illustration to the proof of Lemma 4.3. In the figure $\ell = 5$. On the left, we illustrate the proof by contradiction that induction can be applied to G_v. On the right, we illustrate an elimination tree that is build in the proof.

In the second part of this section, we prove Theorem 1.24. The proof is quite similar to the second part of the proof of Theorem 1.23 in terms of structure and content. Again, we first prove a technical lemma and then derive the theorem.

Lemma 4.4. *Let X be a fan with at least one vertex. Let G be a connected graph and let u be a vertex of G. If G is X-minor-free, then G has a layering $(L_j \mid j \geqslant 0)$ and there is an elimination forest F of $G - u$ with*

(i) $L_0 = \{u\}$, *and*

(ii) $|V(P) \cap L_j| \leqslant \binom{|V(X)|-1}{2}$ *for every root-to-leaf path P in F and for every $j \geqslant 1$.*

Proof. Let x be a vertex of X such that $X - x$ is a path, and let $\ell = |V(X)| - 1$. If $\ell = 0$, then the result is vacuously true, thus, we assume that $\ell > 0$. We proceed by induction on $|V(G)|$. If G has only one vertex, then the result is clear. Hence, assume that G has more vertices.

Let $S = N(u)$ and $G' = G - u$. Observe that G' has no S-rooted model of P_ℓ as otherwise, this model together with a branch set $\{u\}$ added would give a model of X in G. By Theorem 4.2, there is an elimination forest of (G', S') of vertex-height at most $\binom{\ell}{2}$ for some $S' \subseteq V(G')$ containing S. Let S', F' be as above with $S' = V(F')$ of minimum size.

Let C be a component of $G' - V(F')$. We claim that $G - V(C)$ is connected. Suppose to the contrary that there exists a component C' of $G - V(C)$ that does not contain u. In other words, C' is disjoint from $S = N(u)$. Since G is connected, there is an edge vw in G such that $v \in V(C)$ and $w \in V(C')$. More precisely, $w \in V(F')$ since otherwise, C is not a component of $G - V(F')$. It follows that $|S' \setminus V(C')|$ is strictly less than $|S'|$. Let F'' be the forest with vertex set $V(F') \setminus V(C')$, where for all $x, y \in V(F')$, we have $xy \in E(F'')$ whenever x is an ancestor of y or y is an ancestor of x in F', and there is an (x, y)-path in F' with all internal vertices in $V(C')$. For every component C'' of $G' - V(F'')$, either C'' is a component of $G' - V(F')$, or $V(C'') = V(C') \cup V(C)$. Since C' has no neighbors in $V(F'')$, in both cases, there exists a root-to-leaf path in F'' containing the neighborhood of $V(C')$ in G'. Hence F'' is an elimination forest of $(G', S' \setminus V(C'))$. Since vertex-height of F' is at most $\binom{\ell}{2}$ and $|S' \setminus V(C')| < |S'|$, this contradicts the minimality of F'.

Let G_C be obtained from G by contracting $V(G) \setminus V(C)$ into a single vertex u_C, in particular, G_C is a minor of G and therefore G_C is X-minor-free. Since G is connected, S is non-empty, thus, $|V(G_C)| \leqslant |V(G)| - |S \cup \{u\}| + 1 \leqslant |V(G)| - 1$. Hence, by induction hypothesis, there is a layering $(L_{C,j} \mid j \geqslant 0)$ and an elimination forest F_C of $G_C - u_C$ such that

$$L_{C,0} = \{u_C\} \text{ and } |V(P) \cap L_{C,j}| \leqslant \binom{\ell}{2}, \text{ for every root-to-leaf path } P \text{ in } F_C \text{ and } j \geqslant 1.$$

Let $L_0 = \{u\}$, $L_1 = V(F')$, and for every $j \geqslant 2$, $L_j = \bigcup_C L_{C,j-1}$ where C goes over all components of $G' - V(F')$. We claim that $(L_j \mid j \geqslant 0)$ is a layering of G. Indeed, every edge of G is either inside a layer or between two consecutive layers of $(L_j \mid j \geqslant 0)$ since $N(u) = S \subseteq V(F') = L_1$, and $N(V(C)) \subseteq L_1$ and $(L_{C,j} \mid j \geqslant 0)$ is a layering of C, for every component C of $G' - U$.

Let Z be the set of all leaves of F', and for each $y \in Z$, let P_y be the path from the root of F' to y in F'. For every component C of $G' - V(F')$, fix some $\alpha(C) \in Z$ such that the neighborhood of $V(C)$ in G is contained in $P_{\alpha(C)}$. Let F be a forest obtained from F' in the following way. For each component C of $G - V(F')$ add F_C and edges of the form $\alpha(C)x$ for every root x of F_C. Let the roots of F be the roots of F'. It follows that F is an elimination forest of $G - u$.

Finally, let P be a root-to-leaf path in F. We have $|V(P) \cap L_1| \leqslant \binom{\ell}{2}$ since $V(P) \cap L_1$ is the vertex set of a root-to-leaf path of F'. For every $j \geqslant 2$, $V(P) \cap L_j \subseteq V(F_C)$ for some component C of $G' - V(F')$, which implies $|V(P) \cap L_j| = |V(P) \cap L_{C,j-1}| \leqslant \binom{\ell}{2}$. This proves the lemma. □

Proof of Theorem 1.24. Let X be a fan with at least three vertices, and let G be an X-minor-free graph. If G has no vertex, then the result is clear. Hence, we assume that $V(G)$ is non-empty. When G is connected, apply Lemma 4.4 to G with an arbitrary vertex $u \in V(G)$. We obtain an elimination forest F of $G-u$ and a layering $(L_j \mid j \geqslant 0)$ of G such that $|V(P) \cap L_j| \leqslant \binom{|V(X)-1|}{2}$, for every root-to-leaf path P and for every $j \geqslant 1$, and $L_0 = \{u\}$. Let T be obtained by adding u to F as a new root adjacent to all the roots of F. Then T is an elimination tree of G witnessing that $\operatorname{ltd}(G, S) \leqslant \binom{|V(X)|-1}{2}$. When G is not connected, apply the above to each component of G, take for F the disjoint union of the elimination forests obtained for each component, and concatenate the layerings. □

4.3 Excluding a rooted forest and layered pathwidth

Bienstock, Robertson, Seymour, and Thomas [BRST91] first proved that if a graph G has no model of a forest F, then $\operatorname{pw}(G) \leqslant |V(F)| - 2$. Note that the first bound on $\operatorname{pw}(G)$ in terms of $|V(F)|$ was proved by Robertson and Seymour in [RS83]. On the other hand, Diestel gave a beautiful and short proof of the above in [Die95]. We prove an analogous result within the setting of S-rooted models of forests.

Theorem 4.5. *For every forest F with at least one vertex, for every graph G, and for every $S \subseteq V(G)$, if G has no S-rooted model of F, then $\overline{\operatorname{pw}}(G, S) \leqslant 2|V(F)| - 2$.*

The proof of Theorem 4.5 follows Diestel's proof [Die95] that graphs excluding a forest F as a minor have pathwidth at most $|V(F)| - 2$. The notation follows a recent paper by Seymour [Sey23].

Let G be a graph, let w be a positive integer, and let $S \subseteq V(G)$. A separation (A,B) is *(w,S)-good* if it is of order at most w and there is $S' \subseteq V(A)$ containing $S \cap V(A)$ such that (A, S') has a path decomposition of width at most $2w-2$ whose last bag contains $V(A) \cap V(B)$ as a subset. When (A,B), (A',B') are separations of G, we write $(A,B) \leqslant (A',B')$, if $A \subseteq A'$ and $B \supseteq B'$. If moreover the order of (A',B') is at most the order of (A,B), then we say that (A',B') *extends* (A,B). A separation (A,B) in G is *maximal (w,S)-good* if it is (w,S)-good and for every (w,S)-good separation (A',B') extending (A,B), we have $A' = A$ and $B' = B$.

We start with a simple lemma illustrated in Figure 4.2.

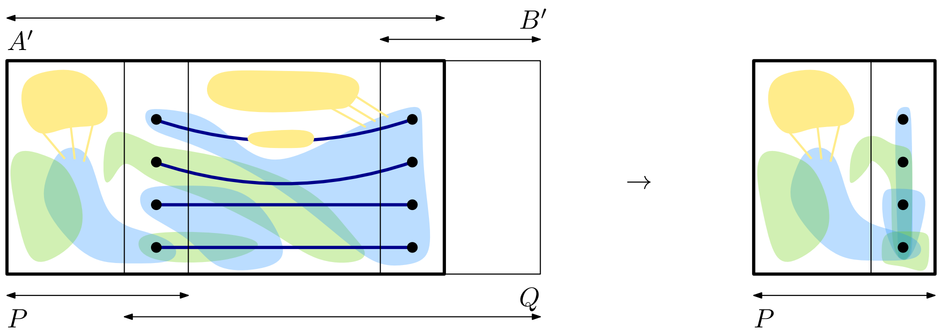


Figure 4.2: Illustration of the proof of Lemma 4.6. On the left, we depict the initial situation and on the right, we depict the result of applying the procedure from the lemma. Bags of path decompositions are green and blue alternately. In yellow, we show the components that are left after removing all vertices of respective path decompositions. The latter is obtained from the former by contracting $|V(P) \cap V(Q)|$ disjoint $(V(P), V(B'))$-paths (in blue).

Lemma 4.6. *Let G be a graph, let w be a positive integer, let $S \subseteq V(G)$, and let (A',B') and (P,Q) be two separations of G with $(P,Q) \leqslant (A',B')$. If (A',B') is (w,S)-good and there are $|V(P) \cap V(Q)|$ vertex-disjoint $(V(P), V(B'))$-paths in G, then (P,Q) is (w,S)-good.*

Proof. Suppose that (A',B') is (w,S)-good and there are $|V(P) \cap V(Q)|$ vertex-disjoint $(V(P), V(B'))$-paths $(R_x \mid x \in V(P) \cap V(Q))$ in G. Suppose $x \in V(R_x)$ for every $x \in V(P) \cap V(Q)$. Let $(W_i \mid i \in [m])$ be a path decomposition of (A', S') for some $S' \subseteq V(A)$ containing $S \cap V(A')$ of width at most $2w-2$ with $V(A') \cap V(B') \subseteq W_m$. Let $(V_i \mid i \in [m])$ be obtained from $(W_i \mid i \in [m])$ by contracting R_x into a single vertex that we identify with x, for every $x \in V(P) \cap V(Q)$. In other words, $V_i = (W_i \cap V(P)) \cup \{x \in V(P) \cap V(Q) \mid V(R_x) \cap W_i \neq \emptyset\}$ for every $i \in [m]$. Observe that $|V_i| \leqslant |W_i|$ for every $i \in [m]$. We claim that $(V_i \mid i \in [m])$ is a path decomposition of (P, S''), where $S'' = \bigcup_{i \in [m]} V_i$, which proves the lemma since $S \cap V(P) \subseteq S''$.

The fact that $(V_i \mid i \in [m])$ is a path decomposition of $H = P[S'']$ follows from the construction. We show that every component C of $P - V(H)$ has its neighborhood in P contained in V_i for some $i \in [m]$. Let $H' = A'[S']$ and let C be a component of $P - V(H)$. Observe that $V(H) \cap V(P) = V(H') \cap V(P)$, and so, $P - V(H)$ is a subgraph of $A' - V(H')$. It follows that C is a connected subgraph of $A' - V(H')$. Therefore, there exists $i \in [m]$ such that the neighborhood of $V(C)$ in A' is contained in W_i, and thus, the neighborhood of C in P is contained in V_i.

Finally, since $V(A') \cap V(B') \subseteq W_m$, we have $V(P) \cap V(Q) \subseteq V_m$. Additionally, $|V(P) \cap V(Q)| = |\{R_x \mid x \in V(P) \cap V(Q)\}| \leqslant |V(A') \cap V(B')| \leqslant w$. All this proves that (P, Q) is (w, S)-good. □

We will use the following version of Menger's theorem in the proof of Lemma 4.8.

Lemma 4.7. *Let G be a graph and let (A, B) and (A', B') be two separations of G. If $(A, B) \leqslant (A', B')$, then there is a separation (P, Q) of G such that*

(i) $(A, B) \leqslant (P, Q) \leqslant (A', B')$*, and*

(ii) *there are $|V(P) \cap V(Q)|$ pairwise disjoint $(V(A), V(B'))$-paths in G.*

Lemma 4.8. *Let w be a positive integer, let G be a graph, and let $S \subseteq V(G)$ such that $\overline{\mathrm{pw}}(G, S) > 2w - 2$. If F is a forest on at most w vertices, then there is a separation (A, B) of G such that*

(s1) $|V(A) \cap V(B)| \leqslant |V(F)|$,

(s2) *there is a $(V(A) \cap V(B))$-rooted model of F in A, and*

(s3) (A, B) *is maximal (w, S)-good.*

Proof. We proceed by induction on $|V(F)|$. Suppose that F is the null graph. Since $(\emptyset, G)$ is a (w, S)-good separation of G, then a maximal (w, S)-good separation (A, B) extending $(\emptyset, G)$ satisfies (s1)-(s3). Next, let F be a non-empty forest on at most w vertices. Let t be a vertex of F of degree at most one.

By the induction hypothesis applied to $F - t$, G has a separation (A^0, B^0) satisfying (s1)-(s3) for $F - t$. Let $(W_i \mid i \in [m])$ be a path decomposition of (A^0, S') of width at most $2w - 2$, for some $S' \subseteq V(A^0)$ containing $S \cap V(A^0)$, with $V(A^0) \cap V(B^0) \subseteq W_m$. If $V(A^0) = V(G)$, then $(W_i \mid i \in [m])$ is a path decomposition of (G, S), which contradicts the fact that $\overline{\mathrm{pw}}(G, S) > 2w - 2$. Hence $V(B^0) \setminus V(A^0) \neq \emptyset$. Let $(B_x \mid x \in V(F - t))$ be a $(V(A^0) \cap V(B^0))$-rooted model of $F - t$ in A^0. If t has degree 0 in F, then choose a vertex $v \in V(B^0) \setminus V(A^0)$ arbitrarily. Otherwise, t has a unique neighbor s in F. Since $|V(A^0) \cap V(B^0)| = |V(F)| - 1$, let u be the unique vertex in $B_s \cap V(A^0) \cap V(B^0)$, and choose v to be a neighbor of u in $V(B^0) \setminus V(A^0)$. Such a neighbor exists as otherwise $(A^0 \cup \{uu' \mid uu' \in E(B^0)\}, B^0 - u)$ is (w, S)-good, which contradicts the maximality of (A^0, B^0).

Let (A, B) be the separation of G defined by $A = G[V(A^0) \cup \{v\}]$ and $B = G[V(B^0)] \setminus E(A)$. Since $V(A^0) \cap V(B^0) \subseteq W_m$, and the neighborhood of v in A is contained in $V(A^0) \cap V(B^0)$, $(W_1, \ldots, W_{m-1}, W_m, V(A) \cap V(B))$ is a path decomposition of $(A, S \cap V(A))$. Moreover, since $|V(A) \cap V(B)| \leqslant |V(F)| \leqslant w \leqslant 2w - 1$, this path decomposition is of order at most $2w - 2$. Therefore, (A, B) is (w, S)-good, and so, there is a maximal (w, S)-good separation (A', B') extending (A, B) in G. In particular, $|V(A') \cap V(B')| \leqslant |V(F)|$.

The next step of the proof is illustrated in Figure 4.3. By Lemma 4.7, there exists a separation (P, Q) such that $(A, B) \leqslant (P, Q) \leqslant (A', B')$, and there is a family $\mathcal{L}$ of $|V(P) \cap V(Q)|$ disjoint $(V(A), V(B'))$-paths in G. If $|V(P) \cap V(Q)| \leqslant |V(F)| - 1$, then by Lemma 4.6, since (A', B') is (w, S)-good, (P, Q) is (w, S)-good as well. Since (P, Q) extends (A^0, B^0), and $v \in V(P) \setminus$

$V(A^0)$, this contradicts the maximality of (A^0, B^0). Hence $|V(P) \cap V(Q)| \geqslant |V(F)|$. Setting $B_t = \{v\}$ gives a $(V(A) \cap V(B))$-rooted model $(B_x \mid x \in V(F))$ of F in A. Since $(A, B) \leqslant (A', B')$, every $(V(A), V(B'))$-path is a $(V(A) \cap V(B))$-$(V(A') \cap V(B'))$-path contained in $V(B) \cap V(A')$. Therefore, the model can be extended using the $|V(F)|$ paths in $\mathcal{L}$ yielding a $(V(A') \cap V(B'))$-rooted model of F in A'. This proves that (A', B') satisfies (s1)-(s3). □

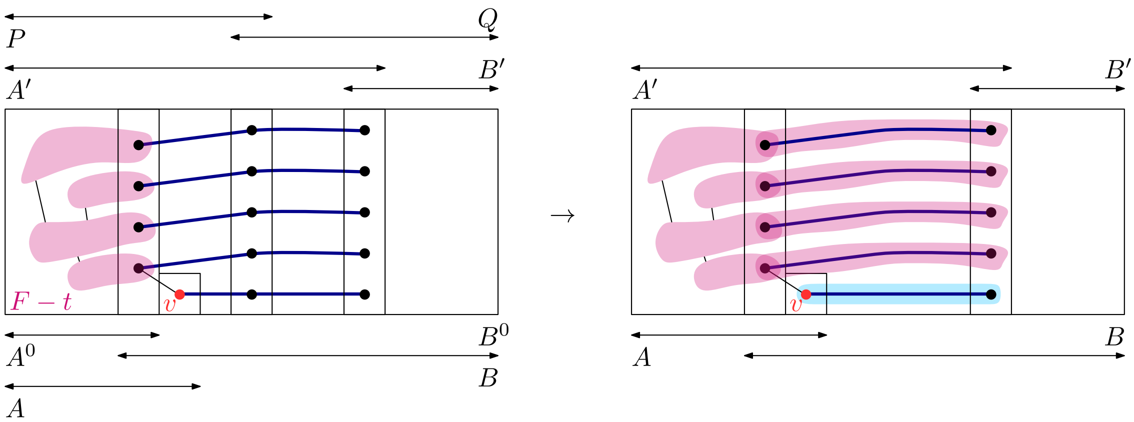


Figure 4.3: An illustration of the proof of Lemma 4.8. We consider F to be a forest ($|V(F)| = 5$ in the figure). In pink, we depict the branch sets of the rooted model of $F - t$. We argue that if $|V(P) \cap V(Q)| < |V(F)|$, then (P, Q) contradicts the maximality of (A^0, B^0). Hence, $V(A)$ is connected with $V(B')$ by 5 pairwise disjoint paths. We add the blue branch set containing v to the model and extend pink branch sets using the paths obtaining a $(V(A') \cap V(B'))$-rooted model of F in A'.

Proof of Theorem 4.5. The proof is illustrated in Figure 4.4. Let F be a forest with at least one vertex, let G be a graph, and let $S \subseteq V(G)$ such that G has no S-rooted model of F. Suppose that $\overline{\mathrm{pw}}(G, S) > 2|V(F)| - 2$ and let $w = |V(F)|$. By Lemma 4.8, G admits a separation (A, B) satisfying (s1)-(s3). Let $(W_i \mid i \in [m])$ be a path decomposition of (A, S') of width at most $2w - 2$, for some $S' \subseteq V(A)$ containing $S \cap V(A)$, with $V(A) \cap V(B) \subseteq W_m$. By Menger's Theorem applied to $V(A)$ and $S \cap V(B)$ there is a separation (P, Q) of G with $V(A) \subseteq V(P)$, $S \cap V(B) \subseteq V(Q)$ and a family $\mathcal{L}$ of $|V(P) \cap V(Q)|$ pairwise disjoint $(V(A), S \cap V(B))$-paths in G.

First, suppose that $|V(P) \cap V(Q)| < |V(F)|$. Then we claim that $(W_1, \dots, W_m, (V(A) \cap V(B)) \cup (V(P) \cap V(Q)))$ is a path decomposition of (P, S'') of width at most $2w - 1$, where $S'' = S' \cup (V(A) \cap V(B)) \cup (V(P) \cap V(Q))$, whose last bag contains $V(P) \cap V(Q)$. Since $S \subseteq S''$, this will contradicts the fact that $\overline{\mathrm{pw}}(G, S) > 2|V(F)| - 2$. Indeed, $|V(A) \cap V(B)| + |V(P) \cap V(Q)| - 1 \leqslant 2w - 2$. Let C be a component of $P - \left(\bigcup_{i \in [m]} W_i \cup (V(P) \cap V(Q))\right)$. Since $V(A) \cap V(B) \subseteq W_m$, either C is a component of $A - \bigcup_{i \in [m]} W_i$ or C is a component of $P - (V(A) \cup (V(P) \cap V(Q)))$. In the former case, there is $i \in [m]$ such that the neighborhood of $V(C)$ in A (and so in P) is contained in W_i. In the latter case, the neighborhood of $V(C)$ in P is contained in $(V(A) \cap V(B)) \cup (V(P) \cap V(Q))$. Hence, (P, Q) is (w, S)-good, which contradicts the maximality of (A, B) because (P, Q) extends (A, B) and $|V(P) \cap V(Q)| < |V(F)| = |V(A) \cap V(B)|$.

It follows that $|V(P) \cap V(Q)| \geqslant |V(F)|$. By (s2), there is a $(V(A) \cap V(B))$-rooted model of F in A. The model combined with the paths in $\mathcal{L}$ yields an S-rooted model of F in G. This contradicts the assumption on G and ends the proof of the theorem. □

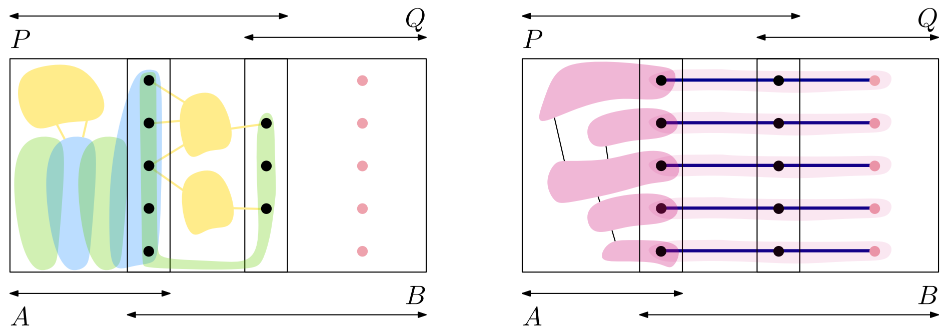


Figure 4.4: An illustration of the proof of Theorem 4.5. On the left, we depict the situation, where $|V(P) \cap V(Q)| < |V(F)|$. We can extend the path decomposition by appending the bag $(V(A) \cap V(B)) \cup (V(P) \cap V(Q))$ (the last green bag in the figure). On the right, we depict the opposite situation, where $|V(P) \cap V(Q)| \geqslant |V(F)|$. Then, we simply extend the model and make it S-rooted.

Finally, we proceed with a proof of Theorem 1.23. The proof is by induction, however, since we need to keep some invariant stronger than the statement of the theorem, we encapsulate it in the following technical lemma, which we later show implies the theorem.

Lemma 4.9. *Let X be an apex-forest with at least two vertices. Let G be a connected graph and let u be a vertex of G. If G is X-minor-free, then G has a layering $(L_j \mid j \geqslant 0)$ and there is a path decomposition $(W_i \mid i \in [m])$ of $G - u$ with*

(i) $L_0 = \{u\}$, *and*

(ii) $|W_i \cap L_j| \leqslant 2|V(X)| - 3$, *for all $i \in [m]$ and $j \geqslant 1$.*

Proof. Let x be a vertex of X such that $X - x$ is a forest, which we denote by F. We proceed by induction on $|V(G)|$. If G has only one vertex, then the result is clear. Hence, assume that G has more vertices.

Let $S = N(u)$ and $G' = G - u$. Observe that G' has no S-rooted model of F, as otherwise, this model together with a branch set $\{u\}$ added would give a model of X in G. By Theorem 4.5, there exist $S' \subseteq V(G')$ containing S, and a path decomposition of (G', S') of width at most $2|V(F)| - 2 = 2|V(X)| - 4$. Let S' be such a set of minimum size, and let $(V_i \mid i \in [m_0])$ be a path decomposition of (G', S') of width at most $2|V(X)| - 4$.

Let C be a component of $G' - S'$. We claim that $G - V(C)$ is connected. Suppose to the contrary that there exists a component C' of $G - V(C)$ that does not contain u. In other words, C' is disjoint from $S = N(u)$. Since G is connected, there is an edge vw in G such that $v \in V(C)$ and $w \in V(C')$. More precisely, $w \in U$ since otherwise C is not a component of $G - S'$. It follows that $S'' = S' - V(C')$ is strictly less than S'. For every component C'' of $G' - S''$, either C'' is a component of $G' - S'$, or $V(C'') = V(C') \cup V(C)$. Since C' has not neighbors in S'', in both cases, there exists $i \in [m_0]$ such that $N(V(C'')) \subseteq V_i \setminus V(C')$. Hence $(V_i \setminus V(C') \mid i \in [m_0])$ is a path decomposition of (G', S''). The width of this path decomposition is at most $2|V(X)| - 4$, which contradicts the minimality of S'.

Let G_C be obtained from G by contracting $V(G) \setminus V(C)$ into a single vertex u_C, in particular, G_C is a minor of G and therefore G_C is X-minor-free. Since G is connected, S is non-empty, thus, $|V(G_C)| \leqslant |V(G)| - |S \cup \{u\}| + 1 \leqslant |V(G)| - 1$. Hence, by induction hypothesis, there is a layering $(L_{C,j} \mid j \geqslant 0)$ and a path decomposition $(V_{C,i} \mid i \in [m_C])$ of $G_C - u_C$ such that

$$L_{C,0} = \{u_C\} \text{ and } |V_{C,i} \cap L_{C,j}| \leqslant 2|V(X)| - 3, \text{ for every } i \in [m_C] \text{ and } j \geqslant 1.$$

Let $L_0 = \{u\}$, $L_1 = S'$, and for every $j \geqslant 2$, $L_j = \bigcup_C L_{C,j-1}$ where C goes over all components of components of $G' - S'$. See Figure 4.5. We claim that $(L_j \mid j \geqslant 0)$ is a layering of G. Indeed, every edge of G is either inside a layer or between two consecutive layers of $(L_j \mid j \geqslant 0)$ since $N(u) = S \subseteq S' = L_1$, and $N(V(C)) \subseteq L_1$ and $(L_{C,j} \mid j \geqslant 0)$ is a layering of C, for every component C of $G' - S'$.

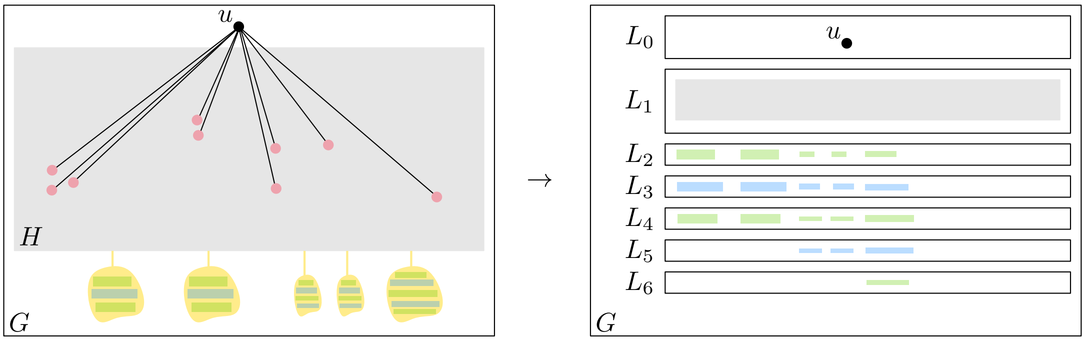


Figure 4.5: An illustration of how we construct the layering $(L_j \mid j \geqslant 0)$ in the proof of Lemma 4.9.

For every component C of $G' - S'$, fix some $\alpha(C) \in [m_0]$ such that the neighborhood of $V(C)$ in G is contained in $V_{\alpha(C)}$. Moreover, let the path decomposition obtained as a concatenation of the path decompositions $(V_{C,i} \mid i \in [m_C])$ for every component C of $G' - U$ with $\alpha(C) = k$ be denoted by $(V_{k,i} \mid i \in [m_k])$ where $m_k = \sum_C m_C$. For every $k \in [m_0]$, let $V'_{k,0} = V_k$ and $V'_{k,i} = V_{k,i} \cup V_k$ for every $i \in [m_k]$. Observe that $(V'_{k,i} \mid 0 \leqslant i \leqslant m_k)$ is a path decomposition of the subgraph of G' induced by $V_k \cup \bigcup_C V(C)$ where C goes over all components of $G' - S'$ with $\alpha(C) = k$. Now, let $(W_i \mid i \in [m])$ be the concatenation of the path decompositions $(V'_{k,i} \mid 0 \leqslant i \leqslant m_k)$ for each $k \in [m_0]$ in the increasing order of k. Here, $m = \sum_{k=1}^{m_0}(m_k + 1)$. This yields a path decomposition of $G - u$. This construction is illustrated in Figure 4.6.

Finally, we argue that the width of $((W_i \mid i \in [m]), (L_j \mid j \geqslant 0))$ is at most $2|V(X)| - 3$. For every $i \in [m]$, we have $W_i \cap L_1 = W_i \cap S' = V_k$ for some $k \in [m_0]$, and so, $|W_i \cap L_1| \leqslant 2|V(X)| - 3$. On the other hand, for every $j \geqslant 2$ and $i \in [m]$, we have $W_i \cap L_j = V_{C,\ell} \cap L_{C,j-1}$ for some component C of $G' - S'$ and $\ell \in [m_C]$, which gives $|W_i \cap L_j| \leqslant 2|V(X)| - 3$ and ends the proof. □

Proof of Theorem 1.23. Let X be an apex-forest with at least two vertices, and let G be an X-minor-free graph. If G has no vertex, then the result is clear. Hence, we assume that $V(G)$ is non-empty. When G is connected, apply Lemma 4.9 to G with an arbitrary vertex $u \in V(G)$. We

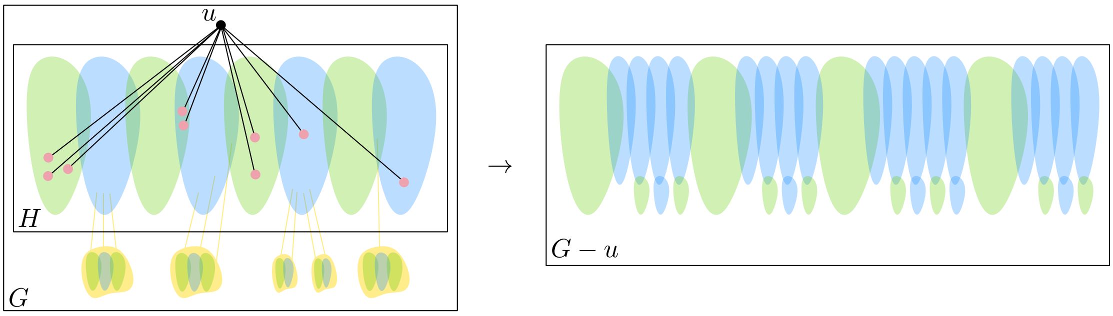


Figure 4.6: An illustration of how we construct the path decomposition $(W_i \mid i \in [m])$ in the proof of Lemma 4.9.

obtain a path decomposition $(W_i \mid i \in [m])$ of $G - u$ and a layering $(L_j \mid j \geqslant 0)$ of G such that $|W_i \cap L_j| \leqslant 2|V(X)| - 3$, for every $i \in [m]$ and $j \geqslant 1$, and $L_0 = \{u\}$. Then $(W_i \cup \{u\} \mid i \in [m])$ is a path decomposition of G such that every bag has intersection with every layer of $(L_j \mid j \geqslant 0)$ of size at most $2|V(X)| - 3$. When G is not connected, apply the above to each component of G and concatenate the layerings and the path decompositions. □

4.4 Excluding a outer-rooted planar graph

The Grid Minor Theorem can be generalized to the setting of S-outer-rooted models as follows, which is equivalent to Theorem 3.11.

Theorem 4.10. *For every plane graph H, there exists a positive integer c_H such that, for every graph G, and for every $S \subseteq V(G)$, if G has no S-outer-rooted model of H, then $\overline{\mathrm{tw}}(G, S) \leqslant c_H$.*

Note that in this result, one cannot replace "S-outer-rooted model" with "S-rooted model". Indeed, for every non-negative integer ℓ, the graph $\boxplus_\ell$ with S_ℓ being the vertex set of the outer face has no S_ℓ-rooted model of K_4, while $\mathrm{tw}(\boxplus_\ell, S_\ell) \geqslant \ell - 1$. The latter inequality follows from Lemma 3.10 applied for the family of all the connected subgraphs that are the union of a row with a column.

It is noteworthy that the proof of Theorem 4.10, together with known results on the Grid-Minor Theorem, actually shows that the constant c_H is at most polynomial in $|V(H)|$. See [HLMR24b] for the details.

We obtain Theorem 4.10 via tangles. First, let us recall the definition of tangles in graphs. Let G be a graph and let k be a positive integer. Let $\mathcal{T}$ be a family of separations of G of order less than k in G. $\mathcal{T}$ is a *tangle* of order k in G if

(T1) for every separation (A, B) of order at most $k - 1$ in G, $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$,

(T2) for every $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$, $A_1 \cup A_2 \cup A_3 \neq G$, and

(T3) for every $(A, B) \in \mathcal{T}$, $V(A) \neq V(G)$.

Marx, Seymour, and Wollan [MSW17] proposed a variant of tangles that is focused on a prescribed set. Additionally, for a fixed $S \subseteq V(G)$, $\mathcal{T}$ is a *tangle* of (G, S) if it is a tangle of G and

(T4) for every $(A,B) \in \mathcal{T}$, $S \not\subseteq V(A)$.

The *tangle number* of (G,S), denoted by $\mathrm{tn}(G,S)$, is the maximum order of a tangle of (G,S). When $S = V(G)$, condition (T4) is vacuous, and $\mathrm{tn}(G) = \mathrm{tn}(G, V(G))$ is the classical tangle number of G. One of the cornerstones of structural graph theory is that the following graph parameters are functionally tied to each other: treewidth, tangle number, and the maximum integer ℓ such that a graph contains a model of $\boxplus_\ell$. An analog in the "focused" setting also holds.

Marx, Seymour, and Wollan [MSW17] showed that the tangle number of (G,S) is functionally tied to the maximum integer ℓ such that G has an S-outer-rooted model of $\boxplus_\ell$ (since this precise statement is not exactly the one showed in [MSW17], a proof of the following is given as an appendix in [HLMR24b]).

Theorem 4.11 ([MSW17])**.** *Let H be a plane graph. There is an integer c_H such that for every graph G, if there is a tangle $\mathcal{T}$ of (G,S) of order c_H, then G contains an S-outer-rooted model of H.*

We prove that $\mathrm{tn}(G,S)$ is functionally tied to $\overline{\mathrm{tw}}(G,S)$.

Theorem 4.12. *For every graph G with at least one vertex, and for every $S \subseteq V(G)$,*

$$\mathrm{tn}(G,S) - 1 \leqslant \overline{\mathrm{tw}}(G,S) \leqslant 10\max\{\mathrm{tn}(G,S), 2\} - 12.$$

Note that Theorem 4.11 and Theorem 4.12 immediately imply Theorem 4.10. In this section, we prove Theorem 4.12. Let G be a graph with at least one vertex, and $S \subseteq V(G)$. Lemma 4.13 directly implies the upper bound, that is, $\mathrm{tw}(G,S) \leqslant 10\max\{\mathrm{tn}(G,S),2\} - 12$. Lemma 4.15 implies the lower bound, that is, $\mathrm{tn}(G,S) - 1 \leqslant \mathrm{tw}(G,S)$.

Lemma 4.13. *Let k be an integer with $k \geqslant 2$. Let G be a graph and let $S \subseteq V(G)$ be such that there is no tangle of (G,S) of order k. Then for every $R \subseteq V(G)$ with $|R| \leqslant 7k-8$, there exists $S' \subseteq V(G)$ containing S, and a tree decomposition $\mathcal{D}$ of (G,S') of width at most $10k-12$ such that there is a bag of $\mathcal{D}$ containing R.*

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| \leqslant 10k-11$, then the tree decomposition consisting of a single bag $V(G)$ witnesses the statement. Thus, we assume that $|V(G)| \geqslant 10k - 10 \geqslant 7k-8$. By possibly adding some vertices to R, we assume without loss of generality that $|R| = 7k-8$.

Let $\mathcal{T}$ be the family of all separations (A,B) of G of order at most $k-1$ such that $|V(A)\cap R| \leqslant 4k-5$. By assumption, $\mathcal{T}$ is not a tangle of (G,S). Therefore, one of (T1)-(T4) does not hold.

If (T1) does not hold, then there is a separation (A,B) of G of order at most $k-1$ such that $|V(A)\cap R| \geqslant 4k-4$ and $|V(B)\cap R| \geqslant 4k-4$. Then $|R| \geqslant |V(A)\cap R| + |V(B)\cap R| - |V(A)\cap V(B)| \geqslant 8k-8-(k-1) = 7k-7 > 7k-8$, a contradiction.

If (T2) does not hold, then there are separations $(A_1,B_1),(A_2,B_2),(A_3,B_3)$ in $\mathcal{T}$ such that $A_1 \cup A_2 \cup A_3 = G$. Let $Z = \bigcup_{i=1}^{3}(V(A_i)\cap V(B_i))$. Let C be a component of $G-Z$, let $G_C = G[V(C)\cup N(C)]$, and let $R_C = N(V(C)) \cup (R \cap V(C))$. Since $V(C) \subseteq V(A_i)$ for some $i \in \{1,2,3\}$, $|V(C)\cap R| \leqslant |V(A_i)\cap R| \leqslant 4k-5$. Note that $V(G_C) = V(C)\cup N(C) \subseteq A_i$, and thus,

$$|V(G)\setminus V(G_C)| \geqslant |V(B_i)\setminus V(A_i)| \geqslant |(V(B_i)\setminus V(A_i))\cap R| \geqslant 3k-3 > 0.$$

Moreover, since $N(V(C)) \subseteq Z$, $|N(V(C))| \leqslant |Z| = 3(k-1)$. Hence, $|R_C| \leqslant |V(C) \cap R| + |N(V(C))| \leqslant 4k-5+3k-3 = 7k-8$. In order to apply induction to G_C and R_C, we have to argue that $|V(G_C)| < |V(G)|$. We have $V(C) \cup N(C) \subseteq A_i$ for some $i \in \{1,2,3\}$. By induction hypothesis applied to G_C and R_C, there is $S'_C \subseteq V(G_C)$ containing $S \cap V(C)$, and a tree decomposition $(T_C, (W_{C,x} \mid x \in V(T_C)))$ of (G_C, S'_C) of width at most $10k-12$ such that $R^C \subseteq W_{C,r_C}$ for some $r_C \in V(T_C)$. Let T be obtained from the disjoint union of T_C for all components C of $G-Z$ by adding a new vertex r and edges rr_C for every component C of $G-Z$. Finally, let $W_r = Z \cup R$, and for every component C of $G-Z$ and every $x \in V(T_C)$, let $W_x = W_{C,x}$. We take $S' = \bigcup_{x\in V(T)} W_x$. Observe that $|W_r| \leqslant |Z|+|R| \leqslant 3(k-1)+7k-8 = 10k-11$. Every component of $G - \bigcup_{x\in V(T)} W_x$ is a subgraph of a component of $G_C - \bigcup_{x\in V(T_C)} W_{C,x}$ for some component C of $G-Z$. Therefore, $(T, (W_x \mid x \in V(T)))$ is a tree decomposition of (G, S') of width at most $10k-12$ such that $R \subseteq W_r$.

If (T3) does not hold, then there is a separation $(A,B) \in \mathcal{T}$ such that $V(A) = V(G)$. It follows that $|R| = |R \cap V(A)| \leqslant 4k-5 < 7k-8 = |R|$, a contradiction.

If (T4) does not hold, then there is a separation $(A,B) \in \mathcal{T}$ such that $S \subseteq V(A)$. Let $R' = (R \cap V(A)) \cup (V(A) \cap V(B))$. Note that $|R'| \leqslant 4k-5+(k-1) = 5k-6 \leqslant 7k-8$. By induction hypothesis applied to A and R', there is $S'_A \subseteq V(A)$ containing $S \cap V(A)$, and a tree decomposition $(T', (W_x \mid x \in V(T')))$ of (A, S'_A) of width at most $10k-12$ such that $R' \subseteq W_{r'}$ for some $r' \in V(T')$. Let T be obtained from T' by adding a new vertex r and the edge rr'. Finally, set $W_r = R \cup (V(A) \cap V(B))$ and observe that $|W_r| \leqslant |R| + k - 1 \leqslant 7k-8+k-1 = 8k-9 \leqslant 10k-12$. Every component of $G - \bigcup_{x\in V(T)} W_x$ is either a component of $B - A$ or is a subgraph of a component of $A - \bigcup_{x\in V(T')} W_x$. In both cases, the neighborhood of the component is contained in a single bag. Therefore, $(T, (W_x \mid x \in V(T)))$ is a tree decomposition of (G, S'), for $S' = \bigcup_{x\in V(T)} W_x = S'_A \cup R \cup (V(A) \cap V(B))$, of width at most $10k-12$ such that $R \subseteq W_r$. Since $S \subseteq S'$, this concludes the lemma. □

In the proof of Lemma 4.15 we use the following simple observation.

Observation 4.14. *Let k be a positive integer, let G be a graph, and let $\mathcal{T}$ be a tangle of G of order k. Let (A,B) and (A',B') be two separations of G of order at most $k-1$ such that $V(A) = V(A')$ and $V(B) = V(B')$. Then,*

$$(B,A) \notin \mathcal{T} \iff (A,B) \in \mathcal{T} \iff (A',B') \in \mathcal{T} \iff (B',A') \notin \mathcal{T}.$$

Proof. The first and last equivalences are clear by (T1) and (T2). In order to prove the middle equivalence, suppose to the contrary that $(A,B) \in \mathcal{T}$ and $(A',B') \notin \mathcal{T}$. By (T2), $(B',A') \in \mathcal{T}$. Observe that $(A \cup B', A' \cap B)$ is a separation of G, and its order is at most $k-1$. By (T3), $(A \cup B', A' \cap B) \notin \mathcal{T}$, hence by (T1), $(A' \cap B, A \cup B') \in \mathcal{T}$. But then $B' \cup A \cup (A' \cap B) = G$, which contradicts (T3). □

Lemma 4.15. *Let k be a positive integer, let G be a graph with at least one vertex, and let $S \subseteq V(G)$. If $\operatorname{tn}(G,S) \geqslant k$, then $\operatorname{tw}(G,S) \geqslant k-1$.*

Proof. Let $\mathcal{T}$ be a tangle of (G,S) of order k. Suppose to the contrary that there is a tree decomposition $(T_0, (W_x \mid x \in V(T_0)))$ of (G,S) of width at most $k-2$. Let $U = \bigcup_{x\in V(T_0)} W_x$. By possibly adding some vertices to T_0 and some bags to $(W_x \mid x \in V(T_0))$, without loss of generality we can assume that every vertex in U is in at least two bags. For every component C of $G-U$, there is a bag $x_C \in V(T_0)$ such that $N(V(C)) \subseteq W_{x_C}$. Let T be obtained from T_0 by adding a new vertex u_C and the edge x_Cu_C for every component C of $G-U$. Let $W_{u_C} = N(V(C)) \cup V(C)$ for every component C of $G-U$. It follows that $(T,(W_x \mid x \in V(T)))$ is a tree decomposition $(T,(W_x \mid x \in V(T)))$ of G. While this tree decomposition may have large width, for every edge $xy \in E(T)$, we have $|W_x \cap W_y| \leqslant k-1$.

Let $Z = V(T) \setminus V(T_0)$ be the set of all added vertices. For every $uv \in E(T)$, recall that $T_{u|v}$ is the component of $T \setminus uv$ containing u, and let $G_{u|v}$ be the subgraph $G[\bigcup_{x \in V(T_{u|v})} W_x]$.

Let $\vec{T}$ be the directed graph with the vertex set $V(T)$ and the arc set consisting of all the pairs $(u,v) \in V(T)^2$ such that $uv \in E(T)$ and for every separation (A,B) of G with $V(A) = V(G_{u|v})$ and $V(B) = V(G_{v|u})$, we have $(A,B) \in \mathcal{T}$. By Observation 4.14, $\vec{T}$ is an orientation of T. Since T is a tree, $\vec{T}$ is acyclic, and thus, there is a sink x in $\vec{T}$. If $x \in Z$, then the neighbor y of x in T is such that $(G_{y|x}, G_{x|y} \setminus E(G_{y|x})) \in \mathcal{T}$, which contradicts (T4) since $S \subseteq V(G_{y|x})$. Hence, $x \notin Z$, and so, $|W_x| \leqslant k-1$. Let $y_1, \ldots, y_d$ be the neighbors of x in T. For every $i \in [d]$, let (A_i, B_i) be a separation of G with $A_i = G\left[\bigcup_{j=1}^{i} V(G_{y_j|x})\right]$ and $B_i = G\left[\bigcap_{j=1}^{i} V(G_{x|y_j})\right] - E(A_i)$. It follows that $V(A_i) \cap V(B_i) \subseteq W_x$. Therefore, (A_i, B_i) has order at most $|W_x| \leqslant k-1$ for every $i \in [d]$.

We claim that $(A_i, B_i) \in \mathcal{T}$ for every $i \in [d]$. We prove this by induction on i. The fact that x is a sink implies that $(A_1, B_1) \in \mathcal{T}$, hence, let $1 < i \leqslant d$, and assume that $(A_{i-1}, B_{i-1}) \in \mathcal{T}$. Suppose to the contrary that $(B_i, A_i) \in \mathcal{T}$. Since $(A_{i-1}, B_{i-1}) \in \mathcal{T}$ and $(G_{y_i|x}, G_{x|y_i} - E(G_{y_i|x})) \in \mathcal{T}$ by (T2), $A_{i-1} \cup G_{y_i|x} \cup B_i \neq G$, which is false since $A_i = A_{i-1} \cup G_{y_i|x}$, and yields $(A_i, B_i) \in \mathcal{T}$.

The above in particular, implies that $(A_d, B_d) \in \mathcal{T}$. However, since every vertex in U is in at least two bags, we have $V(G) = \bigcup_{z \in V(T) \setminus \{x\}} W_z = V(A)$, which contradicts (T3), and shows that there is no tree decomposition of (G,S) of width at most $k-2$. □

4.5 Erdős-Pósa property

In this section, we discuss the applications of our techniques to Erdős-Pósa properties for rooted models. Lemma 3.10 with Theorem 4.10 yield that outer-rooted models of a fixed connected plane graph admit the Erdős-Pósa property.

Corollary 4.16. *For every connected plane graph H, there exists a function $d_H \colon \mathbb{N} \to \mathbb{N}$ such that for every graph G, for every $S \subseteq V(G)$, and for every positive integer k, either*

(i) *G has k vertex-disjoint S-outer-rooted models of H or*

(ii) *there exists a set $Z \subseteq G$ such that $|Z| \leqslant d_H(k-1)$ and $G-Z$ has no S-outer-rooted model of H.*

Proof. Let H be a connected plane graph. For every positive integer k, let $k \cdot H$ denote the plane graph consisting of k disjoint copies of H drawn in the plane in such a way that the outer face of each copy belongs to the outer face. Let $c_{k \cdot H}$ be the constant given by Theorem 4.10 applied to $k \cdot H$. Suppose that G, S, k are like in the statement. Assume that (i) does not hold. In other words, G has no S-outer-rooted model of $k \cdot H$. Therefore, $\overline{\mathrm{tw}}(G,S) \leqslant c_{k \cdot H}$. Then by Lemma 3.10 applied to the family of all the connected subgraphs of G containing an S-outer-rooted model of H, there exists a set Z of at most $(c_{k \cdot H}+1)(k-1)$ vertices in G such that $G-Z$ has no S-outer-rooted-model of H. This proves the corollary for $d_H(k-1) = (c_{k \cdot H}+1)(k-1)$. □

Recently, Dujmović, Joret, Micek, and Morin [DJMM24] showed that for every tree T, for every graph G, for every positive integer k, either G has k disjoint models of T, or there is a set Z of at most $|V(T)|(k-1)$ vertices such that $G - Z$ is T-minor-free. Theorem 4.5 and Lemma 3.10 imply the following Erdős-Pósa property for rooted models of trees.

Corollary 4.17. *For every tree T, for every graph G, for every $S \subseteq V(G)$, and for every positive integer k, either*

(i) *G has k vertex-disjoint S-rooted models of T, or*

(ii) *there exists a set $Z \subseteq G$ such that $|Z| \leqslant (2k|V(T)|-1)(k-1)$ and $G-Z$ has no S-rooted model of T.*

CHAPTER 5

Product structure

The results presented in this chapter are joint work with Vida Dujmović, Robert Hickingbotham, Jędrzej Hodor, Gwenaël Joret, Hoang La, Piotr Micek, Pat Morin, and David R. Wood [DHH$^+$24], but the proof proposed here is new.

This chapter is devoted to the proof of the following theorems, which relies on the notion of rich minors. The ideas of this proof will be extensively used in the next chapter.

Theorem 1.30. *Let X be a nonnull apex graph. There exists a positive integer c such that, for every X-minor-free graph G, there exists a graph H of treewidth at most $2^{\mathrm{td}(X)} - 2$ such that*

$$G \subseteq H \boxtimes P \boxtimes K_c$$

for some path P.

Theorem 1.31. *Let X be a nonnull planar graph. There exists a positive integer c such that the following holds. For every X-minor-free graph G, there exists a graph H of treewidth at most $2^{\mathrm{td}(X)} - 2$ such that*

$$G \subseteq H \boxtimes K_c.$$

Actually, we will prove the following result, which combined with, respectively, Dujmović, Morin, and Wood's result on layered treewidth [DMW17] and the Grid-Minor Theorem (Theorem 1.10) will imply Theorem 1.30 and Theorem 1.31.

Theorem 5.1. *Let X be a nonnull graph. There exists a positive integer c such that the following holds. For every X-minor-free graph G, for every tree decomposition $\mathcal{D}$ of G, there exists a partition $\mathcal{P}$ of $V(G)$ such that*

(A) $\mathrm{tw}(G/\mathcal{P}) \leqslant 2^{\mathrm{td}(X)} - 2$, *and*

(B) *for every $P \in \mathcal{P}$, P is included in a union of at most c bags of $\mathcal{D}$.*

This theorem is inspired by Illingworth, Scott, and Wood [ISW24], who proved a similar result but with $\mathrm{tw}(G/\mathcal{P}) \leqslant |V(X)| - 2$. Their technique is a tool we extensively use in this chapter and the next one.

Proof of Theorem 1.31 assuming Theorem 5.1. By the Grid-Minor Theorem (Theorem 1.10), there exists an integer c_0 such that every X-minor-free graph has treewidth less than c_0. Let G be an X-minor-free graph and let $\mathcal{D}$ be a tree decomposition of G of width less than c_0. Then, by Theorem 5.1, there is a partition $\mathcal{P}$ of $V(G)$ such that

5.1.(A) $\operatorname{tw}(G/\mathcal{P}) \leqslant 2^{\operatorname{td}(X)} - 2$, and

5.1.(B) for every $P \in \mathcal{P}$, P is included in a union of at most c_1 bags of $\mathcal{D}$,

for a positive integer c_1 depending only on X. We take $c = c_0c_1$, and $H = G/\mathcal{P}$. It follows that every part in $\mathcal{P}$ has size at most c, and so for every $P \in \mathcal{P}$, there is an injection ι_P from P to $V(K_c)$. Then the mapping $\varphi\colon V(G) \to V(H \boxtimes K_c)$ defined by $\varphi(u) = (P, \iota_P(u))$ for every $u \in P$, for every $P \in \mathcal{P}$, witnesses the fact that $G \subseteq H \boxtimes K_c$. This proves the theorem. □

Proof of Theorem 1.30 assuming Theorem 5.1. By the aforementioned result of Dujmović, Morin, and Wood in [DMW17], there exists a positive integer c_0 such that every X-minor-free graph has layered treewidth at most c_0. Let G be an X-minor-free graph, and let $(\mathcal{D}, \mathcal{L})$ be a layered tree decomposition of G of width at most c_0. Then, by Theorem 5.1, there is a partition $\mathcal{P}$ of $V(G)$ such that

5.1.(A) $\operatorname{tw}(G/\mathcal{P}) \leqslant 2^{\operatorname{td}(X)} - 2$, and

5.1.(B) for every $P \in \mathcal{P}$, P is included in a union of at most c_1 bags of $\mathcal{D}$,

for a positive integer c_1 depending only on X. Let $(L_i \mid i \in \mathbb{N}) = \mathcal{L}$, and let m be a positive integer such that $L_i = \emptyset$ for every $i > m$. We take $c = c_0c_1$, $H = G/\mathcal{P}$. Let Q be the path with vertices $0, \dots, m$, in this order along Q. For every $P \in \mathcal{P}$, and for every $i \in \{0, \dots, m\}$, we have $|P \cap L_i| \leqslant c_0c_1 = c$ and so there an injection $\iota_{P,i}\colon P \cap L_i \to V(K_c)$. Let $\varphi\colon V(G) \to V(H \boxtimes Q \boxtimes K_c)$ be defined by $\varphi(u) = (P, i, \iota_{P,i}(u))$ for every $u \in P \cap L_i$, for every $P \in \mathcal{P}$, and for every $i \in \{0, \dots, m\}$. Then, φ witnesses the fact that $G \subseteq H \boxtimes Q \boxtimes K_c$, which proves the theorem. □

5.1 Preliminaries

Let T be a tree rooted in a vertex r. For every vertex v of T with $v \neq r$, we define $\mathrm{p}(T, v)$ to be the vertex following v on the path from v to r in T. We call $\mathrm{p}(T, v)$ the *parent* of v in T, and we say that v is a *child* of $\mathrm{p}(T, v)$ in T. For a subtree T' of T, we define $\mathrm{root}(T')$ to be the root of T', that is as the unique vertex in T' closest to r in T.

A *tree partition* of a graph G is a pair $(T, \mathcal{P})$, where T is a rooted tree and $\mathcal{P} = (P_x \mid x \in V(T))$ is a partition of $V(G)$ such that for every edge uv in G either there is $x \in V(T)$ with $u, v \in P_x$, or there is an edge xy in T with $u \in P_x$ and $v \in P_y$. A *tree partition* of (G, S), where G is a graph and $S \subseteq V(G)$, is a tree partition $(T, (P_x \mid x \in V(T)))$ of $G[S]$ such that, for every connected component C of $G - S$, there exists two adjacent or identical vertices x, y in T such that $N_G(V(C)) \subseteq P_x \cup P_y$. Recall that a *path partition* of G (resp. (G, S)) is a tree partition $(T, (P_x \mid x \in V(T)))$ of G (resp. (G, S)) where T is a path.

Let u, v be two (not necessarily distinct) vertices of T. The *lowest common ancestor* of u and v in T, denoted by $\mathrm{lca}(T, u, v)$, is the furthest vertex from the root that has u and v as descendants. Let $Y \subseteq V(T)$. The *lowest common ancestor closure* of Y in T is the set $\mathrm{LCA}(T, Y) = \{\mathrm{lca}(T, u, v) \mid u, v \in Y\}$. See Figure 5.1. Observe that $\mathrm{LCA}(T, \mathrm{LCA}(T, Y)) = \mathrm{LCA}(T, Y)$. The following two lemmas are folklore. See e.g. [DHH+24, Lemma 8] for a proof.

Lemma 5.2. *Let m be a positive integer, let T be a tree, and let Y be a set of m vertices of T. If $X = \mathrm{LCA}(T, Y)$, then $|X| \leqslant 2m - 1$ and for every connected component C of $T - X$, $|N_T(V(C))| \leqslant 2$.*

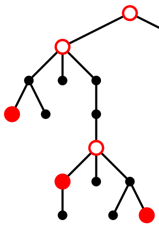


Figure 5.1: For X the set of the full red vertices, $\mathrm{LCA}(T,X)\setminus X$ consists of the red circled vertices.

A tree decomposition $(T,(W_x \mid x\in V(T)))$ of a graph G is *natural* if for every edge xy in T, the graph $G\left[\bigcup_{z\in V(T_{x|y})} W_z\right]$ is connected. The following statement appeared first in [FN06], see also [GJNW23]. A proof of this lemma (in stronger version) will be given in Chapter 6, see Lemma 6.52.

Lemma 5.3 ([FN06, Theorem 1])**.** *Let G be a connected graph and let $(T,(W_x \mid x\in V(T)))$ be a tree decomposition of G. There exists a natural tree decomposition $(T',(W'_x \mid x\in V(T')))$ of G such that for every $x'\in V(T')$ there is $x\in V(T)$ with $W'_{x'}\subseteq W_x$.*

Applying Lemma 5.2 to a natural decomposition, we obtain the following lemma. (See also Figure 5.2.)

Lemma 5.4. *Let m be a positive integer. Let G be a graph and let $\mathcal{W}=(T,(W_x \mid x\in V(T)))$ be a tree decomposition of G. Let X be a set of m vertices of T. Then $Z=\bigcup_{x\in\mathrm{LCA}(T,X)} W_x$ is the union of at most $2m-1$ bags of $\mathcal{W}$ such that for every connected component C of $G-Z$, $N_G(V(C))$ is a subset of the union of at most two bags of $\mathcal{W}$. Moreover, if $\mathcal{W}$ is natural, then $N_G(V(C))$ intersects at most two connected components of $G-V(C)$.*

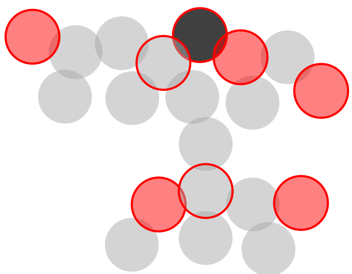


Figure 5.2: Illustration for Lemma 5.4. The bags in X are in red, and Z is the union of the bags encircled in red. The darker bag is the bag of the root. Observe that every connected component of $G-Z$ has neighborhood contained in a union of at most two bags.

For all graphs G_1, G_2, we denote by $G_1\oplus G_2$ the graph obtained from the disjoint union of G_1 and G_2 by adding every edge between $V(G_1)$ and $V(G_2)$. In other words, $G_1\oplus G_2$ is the graph with vertex set $(V(G_1)\times\{1\})\cup(V(G_2)\times\{2\})$ and edge set $\{(u,i)(v,j) \mid i\neq j \text{ or } uv\in E(G_i)\}$.

We define a graph $U_{h,d}$ for all positive integers h,d by induction on h as follows. For every positive integer d,

(i) $U_{1,d}$ consists in d isolated vertices, and

(ii) for every integer h with $h \geqslant 2$, $U_{h,d}$ is the disjoint union of d copies of $K_1 \oplus U_{h-1,d}$.

See Figure 5.3. An immediate induction shows that a graph X has treedepth at most h if and

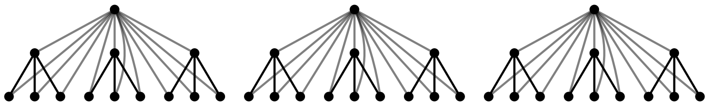


Figure 5.3: The graph $U_{3,3}$. For sake of readability, some edges are depicted in gray.

only if $X \subseteq U_{h,d}$ for some positive integer d. Therefore, it is enough to show Theorem 5.1 for $X = U_{h,d}$.

The following lemma is an important property of the graphs $(U_{h,d} \mid h, d \in \mathbb{N}_{>0})$. Informally, it says that, if d' is large enough compared to k and d, then for every covering of $U_{h,d'}$ by k sets, $U_{h,d'}$ contains a model of $U_{h,d}$ rooted in one of these k sets. Similar results will be extensively used in Chapter 6 under the name "coloring elimination property".

Lemma 5.5. *Let h, d, k be positive integers. There exists a positive integer d' such that for every covering $S_1, \dots, S_k$ of $V(U_{h,d'})$, there exists $i \in [k]$ such that $U_{h,d'}$ contains an S_i-rooted model of $U_{h,d}$.*

Proof. We proceed by induction on h. If $h = 1$, then let $d' = k(d-1)+1$. Let $S_1, \dots, S_k$ be a covering of $V(U_{h,d'})$. By definition, $U_{h,d'}$ consists of d' isolated vertices. By the pigeonhole principle, there exists $i \in [k]$ such that $|S_i| \geqslant d$. Then, for a subset R of S_i of size d, $(\{x\} \mid x \in R)$ is a S_i-rooted model of $U_{h,d}$.

Now suppose $h > 1$ and that the result holds for $h-1$. By the induction hypothesis, there exists a positive integer d'_0 such that for each covering $S_1, \dots, S_k$ of $V(U_{h-1,d'_0})$, there exists $i \in [k]$ such that U_{h-1,d'_0} contains an S_i-rooted model of $U_{h-1,d+1}$.

We take $d' = \max\{kd+1, d'_0\}$. Let $S_1, \dots, S_k$ be a covering of $V(U_{h,d'})$ and let C be a connected component of $U_{h,d'}$. By definition, C is isomorphic to $K_1 \oplus U_{h-1,d'}$. Let r be the vertex corresponding to K_1 in C. Then, by the induction hypothesis, there exists $i_C \in [k]$ and an S_{i_C}-rooted model $\mathcal{M}_{C,0}$ of $U_{h-1,d+1}$ in $C - r$. Let $\mathcal{M}_C$ be the model of $K_1 \oplus U_{h-1,d}$ obtained from $\mathcal{M}_{C,0}$ by taking for the branch set of the vertex of K_1 the union of $\{r\}$ with all the branch set corresponding to the vertices in one of the connected components of $U_{h-1,d+1}$. By construction, $\mathcal{M}_C$ is a S_{i_C}-rooted model of $K_1 \oplus U_{h-1,d}$ in C.

Finally, by the pigeonhole principle, there exists d connected components $C_1, \dots, C_d$ of $U_{h,d'}$ such that

$$i_{C_1} = \cdots = i_{C_d}.$$

Let i be this integer. It follows that the union of $\mathcal{M}_{C_1}, \dots, \mathcal{M}_{C_d}$ is an S_i-rooted model of $U_{h,d}$ in $U_{h,d'}$. This proves the lemma. □

5.2 Proof of Theorem 5.1

Let G be a graph, let $S \subseteq V(G)$, and let $\mathcal{P}$ be a partition of S. If $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ is a tree decomposition of G, then the *$\mathcal{D}$-width* of $\mathcal{P}$ is the smallest integer k such that every part of $\mathcal{P}$ is included in a union of at most k bags of $\mathcal{D}$, i.e. $\forall P \in \mathcal{P}, \exists x_1, \dots, x_k \in V(T), P \subseteq W_{x_1} \cup \dots \cup W_{x_k}$. Recall that a *tree decomposition of* $(G, \mathcal{P})$ is a pair $(T, (W_x \mid x \in V(T)))$ where T is a tree and $W_x \subseteq \mathcal{P}$ for every $x \in V(T)$ such that

(i) for every $u \in S$, $\{x \in V(T) \mid u \in \bigcup W_x\}$ induces a nonempty connected subtree of T,

(ii) for every edge uv of $G[S]$, there exists $x \in V(T)$ such that $u, v \in \bigcup W_x$, and

(iii) for every connected component C of $G - S$, there exists $x \in V(T)$ such that $N_G(V(C)) \subseteq \bigcup W_x$.

The *width* of this tree decomposition is then $\max_{x \in V(T)} |W_x| - 1$, and the *treewidth of* $(G, \mathcal{P})$, denoted by $\mathrm{tw}(G, \mathcal{P})$, is the minimum width of a tree decomposition of $(G, \mathcal{P})$.

Theorem 5.1 will be a consequence of the following technical statement.

Theorem 5.6. *Let h, d be positive integers with $d \geqslant 2$. There is an integer $c_{5.6}(h, d)$ such that, for every graph G, for every tree decomposition $\mathcal{D}$ of G, for every family $\mathcal{F}$ of connected subgraphs of G, if there is no $\mathcal{F}$-rich model of $U_{h,d}$ in G, then there exists $S \subseteq V(G)$ and a partition $\mathcal{P}$ of S such that*

(A) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,*

(B) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $2^{h+1} - 2$ connected components of $G - V(C)$,*

(C) *$\mathcal{P}$ has $\mathcal{D}$-width at most $c_{5.6}(h, d)$, and*

(D) $\mathrm{tw}(G, \mathcal{P}) < 2^h - 1$.

Theorem 5.1 is now a direct application of Theorem 5.6.

Proof of Theorem 5.1 assuming Theorem 5.6. Let $h = \mathrm{td}(X)$, and let d be a positive integer such that $X \subseteq U_{h,d}$. We take $c = c_{5.6}(h, d)$. Let G be an X-minor-free graph, and let $\mathcal{D}$ be a tree decomposition of G. We take $\mathcal{F}$ to be the family of all the connected subgraphs of G. Then, G has no $\mathcal{F}$-rich model of $U_{h,d}$. Therefore, by Theorem 5.6, there exists $S \subseteq V(G)$ and a partition $\mathcal{P}$ of S such that

5.6.(A) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,

5.6.(B) for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $2^{h+1} - 2$ connected components of $G - V(C)$,

5.6.(C) $\mathcal{P}$ has $\mathcal{D}$-width at most c, and

5.6.(D) $\mathrm{tw}(G, \mathcal{P}) < 2^h - 1$.

By the definition of $\mathcal{F}$, 5.6.(A) implies that $S = V(G)$. In particular, $\mathrm{tw}(G, \mathcal{P}) = \mathrm{tw}(G/\mathcal{P}) \leqslant 2^h - 2$ by 5.6.(D). Moreover, since $\mathcal{P}$ has $\mathcal{D}$-width at most c, every part of $\mathcal{P}$ is included is a union of at most c bags of $\mathcal{D}$. This proves Theorem 5.1. □

Proof of Theorem 5.6. We proceed by induction on h. First suppose that $h = 1$. Note that $U_{1,d}$ has d vertices and no edges. We set

$$c_{5.6}(1, d) = 2d - 3.$$

Let G be a graph, let $\mathcal{D}$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of $U_{h,d}$ in G. If $\mathcal{F}$ is empty, then the result is clear for $S = \emptyset$ and $\mathcal{P} = \emptyset$. Now assume $\mathcal{F} \neq \emptyset$.

By Lemma 5.3, we assume that $\mathcal{D}$ is natural. By Lemma 1.17, there is a set which is the union of $d - 1$ bags of $\mathcal{D}$ intersecting every member of $\mathcal{P}$. By Lemma 5.4, this implies that there is a set S which is a union of at most $2(d - 1) - 1 = 2d - 3$ bags such that every member of $\mathcal{F}$ intersects S, and for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $2 = 2^2 - 2$ connected components of $G - V(C)$. We take $\mathcal{P} = \{S\}$. Recall that $\mathcal{F} \neq \emptyset$, and so $\mathcal{P}$ is a partition of S. By the definition of S, the items (A), (B) and (C) hold. Moreover, $\operatorname{tw}(G, \mathcal{P}) \leqslant |\mathcal{P}| = 1 = 2^1 - 1$, and so (D) holds. This concludes the case $h = 1$.

Now suppose $h > 1$ and that the result holds for $h - 1$. We start with the following claim, which can be seen as an analog of Lemma 3.14. (See also Figure 5.4.)

Claim 5.6.1. *Let d be a positive integer. There exists an integer $c_{5.6.1}(h, d)$ such that, for every connected graph G, for every family $\mathcal{F}$ of connected subgraphs of G, for every nonempty set $R \subseteq V(G)$ of size at most $2^h - 2$, for every tree decomposition $\mathcal{D}$ of $G - R$, if there is no $\mathcal{F}$-rich model of $K_1 \oplus U_{h-1,d}$ in G, then there exists $S \subseteq V(G)$, and a tree partition $(A, (P_y \mid y \in V(A)))$ of (G, S) with A rooted in $a \in V(A)$ such that*

(a) $R = P_a$;

(b) $V(F) \cap S \neq \emptyset$ *for every* $F \in \mathcal{F}$;

(c) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $2^{h+1} - 4$ connected components of $G - V(C)$; and*

(d) *for every $y \in V(A) \setminus \{a\}$, there exists a partition $\mathcal{P}_y$ of P_y such that*

    (i) $\operatorname{tw}(G_y, \mathcal{P}_y) < 2^{h-1} - 1$, *and*

    (ii) *$\mathcal{P}_y$ has $\mathcal{D}$-width at most $c_{5.6.1}(h, d)$,*

where G_y is the subgraph of G induced by the union of $U_y = \bigcup_{z \in V(A_y)} P_z$[1] *with $V(C)$ for all the connected components C of $G - S$ such that $N_G(V(C)) \cap U_y \neq \emptyset$.*

Proof of the claim. Let $k = 2^h - 2$. By Lemma 5.5 there exists a positive integer d' such that for every covering $S_1, \ldots, S_k$ of $V(U_{h-1,d'})$, there exists $i \in [k]$ such that $U_{h-1,d'}$ contains an S_i-rooted model of $U_{h-1,d+1}$. Let

$$c_{5.6.1}(h, d) = c_{5.6}(h - 1, d').$$

Let G be a connected graph, let $\mathcal{D}$ be a tree decomposition of G, let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of $K_1 \oplus U_{h-1,d+1}$ in G, and let $R \subseteq V(G)$ be nonempty and of size at most $2^h - 2$.

[1] Recall that for a rooted tree A and for $y \in V(A)$, we denote by A_y the subtree of A rooted in y.

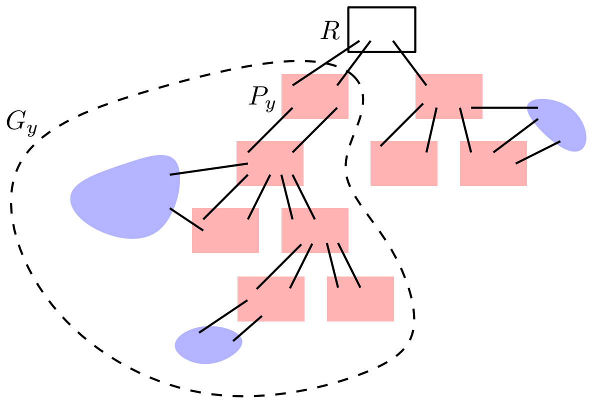


Figure 5.4: Illustration for Claim 5.6.1. When exlcuding $K_1 \oplus U_{h-1,d}$, we obtain a tree partition where every part has, in its own subtree, the structure given by the induction hypothesis (see Item (d)).

We proceed by induction on $|V(G-R)|$. If $\mathcal{F}|_{G-R} = \emptyset$, then the result is clear for $S = R$, $V(A) = \{a\}$, and $\mathcal{P} = \{R\}$. Now suppose that $\mathcal{F}|_{G-R} \neq \emptyset$, and so in particular $V(G-R) \neq \emptyset$.

Let $\mathcal{F}'$ be the family of all the connected subgraphs H of $G-R$ such that there exists $F \in \mathcal{F}$ with $F \subseteq H$, and $N_G(R) \cap V(H) \neq \emptyset$. We claim that there is no $\mathcal{F}'$-rich model of $U_{h-1,d'}$ in $G-R$. Suppose for contradiction that there is an $\mathcal{F}'$-rich model $(B_x \mid x \in V(U_{h-1,d'}))$ of $U_{h-1,d'}$ in $G-R$. For every $r \in R$, let $S_r = \{x \in V(U_{h-1,d'}) \mid r \in N_G(B_x)\}$. Since $(B_x \mid x \in V(U_{h-1,d'}))$ is $\mathcal{F}'$-rich, the family $(S_r)_{r \in R}$ is a covering of $V(U_{h-1,d'})$. Therefore, by Lemma 5.5 and the definition of d', there exists $r \in R$ and a S_r-rooted model $(C_y \mid y \in V(U_{h-1,d+1}))$ of $U_{h-1,d+1}$ in $U_{h-1,d'}$. For every $y \in U_{h-1,d+1}$, let $D_y = \bigcup_{x \in C_y} B_x$. Then, $(D_y \mid y \in V(U_{h-1,d+1}))$ is an $\mathcal{F}$-rich model of $U_{h-1,d+1}$ such that for every $y \in V(U_{h-1,d+1})$, $r \in N_G(D_y)$. Let B be a connected component of $U_{h-1,d+1}$. Note that $U_{h-1,d+1} - B$ is isomorphic to $U_{h-1,d}$. Then the union of $(D_y \mid y \in V(U_{h-1,d+1} - V(B)))$ with the branch set $\{r\} \cup \bigcup_{y \in V(B)} D_y$ is an $\mathcal{F}$-rich model of $K_1 \oplus U_{h-1,d}$ in G, a contradiction. This proves that $G-R$ has no $\mathcal{F}'$-rich model of $U_{h-1,d'}$.

Hence, by the induction hypothesis of the theorem, there exists $S_0 \subseteq V(G-R)$ and a partition $\mathcal{P}_0$ of S_0 such that

(A') $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}'$,

(B') for every connected component C of $G-(R \cup S_0)$, $N_{G-R}(V(C))$ intersects at most $2^h - 2$ connected components of $G-(R \cup V(C))$,

(C') $\mathcal{P}_0$ has $\mathcal{D}$-width at most $c_{5.6}(h-1, d')$, and

(D') $\mathrm{tw}(G-R, \mathcal{P}_0) < 2^{h-1} - 1$.

Note that since $\mathcal{F}|_{G-R} \neq \emptyset$ and G is connected, we have $\mathcal{F}' \neq \emptyset$, and so $S_0 \neq \emptyset$. Let $\mathcal{C}$ be the family of all the connected components C of $G-(R \cup S_0)$ such that $\mathcal{F}|_C \neq \emptyset$.

Let $C \in \mathcal{C}$. Since C is disjoint from S_0 and contains a member of $\mathcal{F}$, by (A'), we have $N_G(V(C)) \cap R = \emptyset$. Moreover, by (B') and because G is connected, $G - V(C)$ has at most $2^h - 2$

connected components. Let G_C be the graph obtained from G by contracting all the connected components of $G - V(C)$ into single vertices, and let R_C be the set of the vertices resulting from these contractions. Note that $|R_C| \leqslant 2^h - 2$. Since $S_0 \neq \emptyset$, we have $|V(G_C - R_C)| = |V(C)| < |V(G - R)|$, and so by the induction hypothesis applied to $G_C, \mathcal{D}|_C, \mathcal{F}|_C$, and R_C, there exists $S_C \subseteq V(G_C)$, and a tree partition $(A_C, (P_{C,y} \mid y \in V(A)))$ with A_C rooted in $a_C \in V(A_C)$ such that

(a') $R_C = P_{C,a_C}$

(b') $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}_C$,

(c') for every connected component C' of $G_C - S_C$, $N_{G_C}(V(C'))$ intersects at most $2^{h+1} - 4$ connected components of $G - V(C)$, and

(d') for every $y \in V(A_C) \setminus \{a_C\}$, there exists a partition $\mathcal{P}_{C,y}$ of $P_{C,y}$ such that

  (i) $\mathrm{tw}(G_{C,y}, \mathcal{P}_{C,y}) < 2^{h-1} - 1$, and

  (ii) $\mathcal{P}_{C,y}$ has $\mathcal{D}|_C$-width at most $c_{5.6.1}(h, d)$,

where $G_{C,y}$ is the subgraph of G_C induced by the union of $U_{C,y} = \bigcup_{z \in V(A_{C,y})} P_{C,z}$ with $V(C')$ for all the connected components C' of $G_C - S_C$ such that $N_{G_C}(V(C')) \subseteq U_{C,y}$. We fix such a partition $\mathcal{P}_{C,y}$.

We suppose that the trees A_C for $C \in \mathcal{C}$ have disjoint vertex sets, and that a, a_0 are fresh vertices.

Let A be the tree rooted in a defined by

$$V(A) = \{a, a_0\} \cup \bigcup_{C \in \mathcal{C}} (V(A_C) \setminus \{a_C\})$$

and

$$E(A) = \{aa_0\} \cup \bigcup_{C \in \mathcal{C}} (E(A_C - a_C) \cup \{a_0 y \mid y \in N_{A_C}(a_C)\}).$$

Then, let

$$P_a = R, \quad P_{a_0} = S_0, \quad \mathcal{P}_{a_0} = \mathcal{P}_0,$$

and for every $C \in \mathcal{C}$, for every $y \in V(A_C - a_C)$, let

$$P_y = P_{C,y}, \quad \mathcal{P}_y = \mathcal{P}_{C,y}.$$

Finally, let

$$S = R \cup S_0 \cup \bigcup_{C \in \mathcal{C}} (S_C \setminus R_C).$$

Note that, as wanted, $\{P_x \mid x \in V(A)\}$ is a partition of S.

We claim that $(A, (P_x \mid x \in V(A)))$ is a tree partition of (G, S). Let uv be an edge of $G[S]$. We want to show that there exists $y, z \in V(A)$ adjacent or identical such that $u \in A_y$ and $v \in A_z$. If $u \in R$, then $v \notin \bigcup_{C \in \mathcal{C}} V(C)$ by the definition of $\mathcal{C}$. Hence, $v \in R \cup S_0 = P_a \cup P_{a_0}$. Now suppose that $u, v \notin R$. If $u, v \in S_0$, then we are done. Otherwise, without loss of generality,

$v \in V(C)$ for some $C \in \mathcal{C}$. If $u \in S_0$, then let $r \in R_C$ be the vertex in G_C resulting from the contraction of the connected component of u in $G - V(C)$. Then, rv is an edge in $G_C[S_C]$. Since $(A_C, (P_{C,y} \mid y \in V(A_C)))$ is a tree partition of (G_C, S_C), and because $P_{C,a_C} = R_C$, there exists $y \in N_{A_C}(a_C)$ such that $v \in P_{C,y}$. It follows that $v \in P_y$, $u \in P_{a_0}$, and $a_0y \in E(A)$. Finally, suppose $u \notin S_0$, and so $u, v \in V(C)$. Then uv is an edge of G_C, and so there exist $y, z \in E(A_C)$ adjacent or identical such that $u \in P_{C,y}$ and $v \in P_{C,z}$. Moreover, since $u, v \notin R_C$, $y, z \neq a_C$. Hence y and z are either adjacent or identical vertices of A, and $u \in P_y, v \in P_z$.

Let C' be a connected component of $G - S$. Let C be the connected component of $G-(R \cup S_0)$ containing C'. We want to show that there exists $y, z \in V(A)$ adjacent or identical such that $N_G(V(C')) \subseteq P_y \cup P_z$. If $C \notin \mathcal{C}$, then $C = C'$, and so $N_G(V(C')) = N_G(V(C)) \subseteq R \cup S_0 = P_a \cup P_{a_0}$ and we are done. Now suppose $C \in \mathcal{C}$. Then C' is a connected component of $G_C - S_C$. Since $(A_C, (P_{C,y} \mid y \in V(A_C)))$ is a tree partition of (G_C, S_C), there exists $y, z \in V(A_C)$ adjacent or identical such that $N_{G_C}(V(C')) \subseteq P_{C,y} \cup P_{C,z}$. If $y, z \neq a_C$, then $N_G(V(C')) = N_{G_C}(V(C')) \subseteq P_{C,y} \cup P_{C,z} = P_y \cup P_z$ and we are done. If $y = a_C$ and $z \neq a_C$, then $N_G(V(C)) \subseteq S_0 \cup P_{C,z} = P_{a_0} \cup P_z$. Finally, if $x = y = a_C$, then $N_G(V(C)) \subseteq S_0 = P_{a_0}$. This shows that $(A, (P_x \mid x \in V(A)))$ is a tree partition of (G, S).

We now prove (a) to (d). Item (a) holds by construction. To show (b), consider $F \in \mathcal{F}$. If F intersects $R \cup S_0$, then $V(F) \cap S \neq \emptyset$. Otherwise, let C be the connected component of $G - (R \cup S_0)$ containing F. Since $F \in \mathcal{F}|_C$, we have $C \in \mathcal{C}$. Then by (b'), $V(F) \cap S_C \neq \emptyset$. Since $V(F)$ is disjoint from R_C, we conclude that $V(F) \cap S \neq \emptyset$. This proves (b).

Let C' be a connected component of $G - S$. We want to show that $N_G(V(C'))$ intersects at most $2^{h+1} - 4$ connected components of $G - V(C')$. Let C be the connected component of $G-(R \cup S_0)$ containing C'. If $C \notin \mathcal{C}$, then $C = C'$. In particular, $N_G(V(C')) \subseteq R \cup S_0$. Then, by (B'), $N_{G-R}(V(C))$ intersects at most $2^h - 2$ connected components of $G-(R \cup V(C))$. Moreover, $|R| \leqslant 2^h - 2$. We deduce that $N_G(V(C'))$ intersects at most $(2^h - 2) + (2^h - 2) = 2^{h+1} - 4$ connected components of $G - V(C')$. Now assume that $C \in \mathcal{C}$. Then C' is a connected component of $G_C - S_C$, and by (c'), $N_{G_C}(V(C'))$ intersects at most $2^{h+1} - 4$ connected components of $G_C - V(C')$. Since G_C was obtained from G by contracting some connected subgraphs, we deduce that $N_G(V(C'))$ intersects at most $2^{h+1} - 4$ connected components of $G - V(C')$. This proves (c).

Let $y \in V(A) \setminus \{a\}$, let $U_y = \bigcup_{z \in V(A_y)} P_z$, and let G_y be the subgraph of G induced by the union of U_y with all the connected components C of $G - S$ such that $N_G(V(C)) \cap U_y \neq \emptyset$. If $y = a_0$, then $G_y \subseteq G - R$, and so by (D'), $\mathrm{tw}(G - R, \mathcal{P}_{a_0}) < 2^{h-1} - 1$ and $\mathcal{P}_{a_0}$ has $\mathcal{D}$-width at most $c_{5.6}(h-1, d') \leqslant c_{5.6.1}(h, d)$ by (C'). Now suppose that $y \neq a_0$. Hence there exists $C \in \mathcal{C}$ such that $y \in V(A_C) \setminus \{a_C\}$. Then $G_y = G_{C,y}$, and so by (d'), $\mathrm{tw}(G_y, \mathcal{P}_y) < 2^{h-1} - 1$ and $\mathcal{P}_y$ has $\mathcal{D}|_C$-width, and so $\mathcal{D}$-width, at most $c_{5.6}(h-1, d)$. This proves (d), and concludes the proof of the claim. $\Diamond$

We can now deduce the following claim, which corresponds to excluding $K_1 \oplus U_{h-1,d}$ instead of $U_{h,d}$ in the theorem. The proof consists in applying Claim 5.6.1, and combining the obtained small tree decompositions of $(G_y, \mathcal{P}_y)$ into a tree decomposition of $(G, \mathcal{P})$ for a suitable $\mathcal{P}$.

Claim 5.6.2. *Let d be a positive integer. For every graph G, for every tree decomposition $\mathcal{D}$ of G, for every family $\mathcal{F}$ of connected subgraphs of G, if there is no $\mathcal{F}$-rich model of $K_1 \oplus U_{h-1,d}$ in G, then there exists $S \subseteq V(G)$ and a partition $\mathcal{P}$ of S such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,*

(b) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $2^{h+1} - 4$ connected components of $G - V(C)$,*

(c) *$\mathcal{P}$ has $\mathcal{D}$-width at most $c_{5.6.1}(h,d)$, and*

(d) $\operatorname{tw}(G, \mathcal{P}) < 2^h - 2$.

Proof of the claim. Let G be a graph, let $\mathcal{D} = \big(T, (W_x \mid x \in V(T))\big)$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $K_1 \oplus U_{h-1,d}$.

First suppose that G is connected. By Claim 5.6.1 applied for $R \subseteq V(G)$ being an arbitrary singleton, there exists $S \subseteq V(G)$, a tree partition $\big(A, (P_y \mid y \in V(A))\big)$ of (G, S) with A rooted in a such that

5.6.1.(a) $R = P_a$

5.6.1.(b) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,

5.6.1.(c) for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $2^{h+1} - 4$ connected components of $G - V(C)$, and

5.6.1.(d) for every $y \in V(A) \setminus \{a\}$, there exists a partition $\mathcal{P}_y$ of P_y such that

(i) $\operatorname{tw}(G_y, \mathcal{P}_y) < 2^{h-1} - 1$, and

(ii) $\mathcal{P}_y$ has $\mathcal{D}$-width at most $c_{5.6.1}(h,d)$,

where G_y is the subgraph of G induced by the union of $U_y = \bigcup_{z \in V(A_y)} P_z$ with $V(C)$ for every connected component C of $G - S$ such that $N_G(V(C)) \cap U_y \neq \emptyset$. We fix such a partition $\mathcal{P}_y$.

Suppose that $A, (P_y \mid y \in V(A)), (\mathcal{P}_y \mid y \in V(A) \setminus \{a\})$ are as above and with $|V(A)|$ is maximal. This is well defined because $P_y \neq \emptyset$ for every $y \in V(A)$, and so $|V(A)| \leqslant |V(G)|$.

We claim that G_y is connected for every $y \in V(A) \setminus \{a\}$. Suppose otherwise, and consider a vertex $y \in V(A) \setminus \{a\}$ with G_y not connected. Let $\mathcal{C}$ be the family of all the connected components of G_y. Let A' be the tree rooted in a defined by

$$\begin{aligned} V(A') &= (V(A) \setminus V(A_y)) \cup \{(z, C) \mid z \in V(A_y), P_z \cap V(C) \neq \emptyset\} \\ E(A') &= E(A - V(A_y)) \\ &\qquad \cup \{(z,C)(z',C) \mid zz' \in E(A_y), C \in \mathcal{C}, P_z \cap V(C) \neq \emptyset, P_{z'} \cap V(C) \neq \emptyset\} \\ &\qquad \cup \{\mathrm{p}(A,y)(y,C) \mid C \in \mathcal{C}, P_y \cap V(C) \neq \emptyset\}. \end{aligned}$$

Less formally, we split A_y in $|\mathcal{C}|$ copies, and we remove the vertices in the resulting trees corresponding to empty parts. One can check that, because G is connected, A is indeed a tree. Then, for every $z \in V(A) \setminus V(A_y)$, let

$$P'_z = P_z, \quad \text{and} \quad \mathcal{P}'_z = \mathcal{P}_z \text{ if } z \neq a,$$

and for every $z \in V(A_y)$, for every $C \in \mathcal{C}$ with $P_z \cap V(C) \neq \emptyset$, let

$$P'_{(z,C)} = P_z \cap V(C), \quad \text{and} \quad \mathcal{P}'_{(z,C)} = \{U \cap V(C) \mid U \in \mathcal{P}_y, U \cap V(C) \neq \emptyset\}.$$

It is then straightforward to show that $A', (P'_z \mid z \in V(A')), (\mathcal{P}'_z \mid z \in V(A') \setminus \{a\})$ satisfy 5.6.1.(a) to 5.6.1.(d). Moreover, for every $z \in V(A')$, there exists $f(z) \in V(A)$ such that $P'_z \subseteq P_{f(z)}$. Since $P_y \neq \emptyset$ for every $y \in V(A)$, $f\colon V(A') \to V(A)$ is surjective. Additionally, because G is connected, there exists $C, C' \in \mathcal{C}$ distinct such that $V(C) \cap P_y \neq \emptyset$ and $V(C') \cap P_y \neq \emptyset$, which implies $(y, C), (y, C') \in V(A')$ and $f((y, C)) = y = f((y, C'))$. Therefore, f is surjective but not injective, and so $|V(A')| > |V(A)|$, which contradicts the choice of $A, (P_y \mid y \in V(A)), (\mathcal{P}_y \mid y \in V(A) \setminus \{a\})$. Therefore, G_y is connected for every $y \in V(A) \setminus \{a\}$.

Let

$$\mathcal{P} = \{R\} \cup \bigcup_{y \in V(A-a)} \mathcal{P}_y.$$

Items 5.6.1.(b) and 5.6.1.(c) directly imply (a) and (b). Moreover, (c) follows from 5.6.1.(d) and the fact that R is a singleton.

We now show (d). Let $y \in V(A)$. If $y \neq a$, then by 5.6.1.(d), $\operatorname{tw}(G_y, \mathcal{P}_y) < 2^{h-1} - 1$. If $y = a$, then $\operatorname{tw}(G, \mathcal{P}_a) < 1 \leqslant 2^{h-1} - 1$ as well since $S_a = R$ and $\mathcal{P}_a = \{R\}$. In both cases, let $(T_y, (W_{y,z} \mid z \in V(T_a)))$ be a tree decomposition of $(G_y, \mathcal{P}_y)$ of width less than $2^{h-1} - 1$.

Let $y \in V(A) \setminus \{a\}$, and let y' be the parent of y in A. Pick $s_y \in V(T_y)$ arbitrarily. Since $(T_{y'}, (W_{y',z} \mid z \in V(T_{y'})))$ is a tree decomposition of $(G_{y'}, \mathcal{P}_{y'})$, and because G_y is connected, there exists $z_y \in V(T_{y'})$ such that $N_G(V(G_y)) \subseteq \bigcup W_{y',z_y}$. Then for every $z \in V(T_y)$, let

$$W_z = W_{y,z} \cup W_{y',z_y}.$$

Moreover, for every $z \in V(T_a)$, let $W_z = W_{a,z}$. We assume that the trees T_y for $y \in V(A)$ have pairwise disjoint vertex sets.

Finally, let T be the tree defined by

$$V(T) = \bigcup_{y \in V(A)} V(T_y)$$

and

$$E(T) = E(T_a) \cup \bigcup_{y \in V(A) \setminus \{a\}} (E(T_y) \cup \{s_y z_y\}).$$

We claim that $(T, (W_x \mid x \in V(T)))$ is a tree decomposition of $(G, \mathcal{P})$. First, we show that for every $u \in S$, the set $\{x \in V(T) \mid u \in \bigcup W_x\}$ induces a nonempty connected subtree of T. Let $u \in S$ and let $y' \in V(A)$ such that $u \in P_{y'} = \bigcup \mathcal{P}_{y'}$. Let X be the set $\{z \in V(T) \mid u \in \bigcup W_z\}$. Since $(T_y, (W_{y',z} \mid z \in V(T_{y'})))$ is a tree decomposition of $(G_{y'}, \mathcal{P}_{y'})$, the set $X_0 = \{z \in V(T_{y'}) \mid u \in \bigcup W_{y',z}\}$ induces a nonempty subtree of $T_{y'}$. Since $X_0 \subseteq X$, this implies that X is nonempty. Now, every vertex $x \in X \setminus X_0$ belongs to $V(T_y)$ for some child y of y' in A such that $u \in \bigcup W_{y',z_y}$, which implies $z_y \in X_0$. Therefore,

$$X = X_0 \cup \bigcup_{y \text{ child of } y',\, u \in \bigcup W_{y',z_y}} V(T_y),$$

and for every y child of y' with $u \in \bigcup W_{y',z_y}$, there is an edge in T between X_0 and $V(T_y)$ (namely $s_y z_y$). This implies that X induces a nonempty connected subtree of T.

Second, we show that every edge of $G[S]$ lies in the union of a bag. For every edge uv of $G[S]$, since $(A, (P_x \mid x \in V(A)))$ is a tree partition of (G, S), there exists $y, y' \in V(A)$ identical or adjacent such that $u \in P_y$ and $v \in P_{y'}$. If $y = y'$, then since $(T_y, (W_{y,z} \mid z \in V(T_y)))$ is a tree decomposition of $(G_y, \mathcal{P}_y)$, there exists $z \in V(T_y)$ such that $u, v \in \bigcup W_{y,z}$, and so $u, v \in \bigcup W_z$. Now suppose that $y \neq y'$. Without loss of generality, y' is the parent of y in A. Since $(T_y, (W_{y,z} \mid z \in V(T_y)))$ is a tree decomposition of $(G_y, \mathcal{P}_y)$, there exists $z \in V(T_y)$ such that $v \in \bigcup W_{y,z}$. Then, since $u \in N_G(V(G_y))$, $u \in \bigcup W_{y',z_y}$, and so $u, v \in \bigcup(W_{y,z} \cup W_{y',z_y}) = \bigcup W_y$.

Third, we show that the neighborhood of every connected component of $G - S$ is contained in the union of a bag. For every connected component C of $G - S$, since $(A, (P_x \mid x \in V(A)))$ is a tree partition of (G, S), there exists $y, y' \in V(A)$ identical or adjacent such that $N_G(V(C)) \subseteq P_y \cup P_{y'}$. If $y = y'$, then since $(T_y, (W_{y,z} \mid z \in V(T_y)))$ is a tree decomposition of $(G_y, \mathcal{P}_y)$, there exists $z \in V(T_y)$ such that $N_{G_y}(V(C)) \subseteq \bigcup W_{y,z}$, and since $N_G(V(C)) = N_{G_y}(V(C))$, we conclude that $N_G(V(C)) \subseteq \bigcup W_{y,z} \subseteq \bigcup W_z$. Now suppose that $y \neq y'$. Without loss of generality, y' is the parent of y in A. Since $(T_y, (W_{y,z} \mid z \in V(T_y)))$ is a tree decomposition of $(G_y, \mathcal{P}_y)$, there exists $z \in V(T_y)$ such that $N_{G_y}(V(C)) \subseteq \bigcup W_{y,z}$. Then, since $N_G(V(C)) \subseteq N_{G_y}(V(C)) \cup N_G(V(G_y))$, and $N_G(V(G_y)) \subseteq \bigcup W_{y',z_y}$, we conclude that

$$N_G(V(C)) \subseteq \bigcup(W_{y,z} \cup W_{y',z_y}) = \bigcup W_z.$$

This proves that $(T, (W_x \mid x \in V(T)))$ is a tree decomposition of $(G, \mathcal{P})$.

Finally, by construction, every bag of $(T, (W_x \mid x \in V(T)))$ has size at most $2(2^{h-1} - 1) = 2^h - 2$. This proves (d) and concludes the case G connected.

Now suppose that G is not connected. Let $\mathcal{C}$ be the family of all the connected components of G. Let $C \in \mathcal{C}$. By applying the previous case to $C, \mathcal{D}|_C, \mathcal{F}|_C$, there exists $S_C \subseteq V(C)$ and a partition $\mathcal{P}_C$ of S_C such that

(a') $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}|_C$,

(b') for every connected component C' of $C - S_C$, $N_C(V(C'))$ intersects at most $2^{h+1} - 4$ connected components of $C - V(C')$,

(c') $\mathcal{P}_C$ has $\mathcal{D}$-width at most $c_{5.6.1}(h, d)$, and

(d') $\mathrm{tw}(C, \mathcal{P}_C) < 2^h - 2$.

Let

$$S = \bigcup_{C \in \mathcal{C}} S_C \quad \text{and} \quad \mathcal{P} = \bigcup_{C \in \mathcal{C}} \mathcal{P}_C.$$

Then (a), (b), and (c) directly follow from, respectively, (a'), (b'), and (c'). Finally, it is a consequence of the definition of tree decompositions of $(G, \mathcal{P})$ that

$$\mathrm{tw}(G, \mathcal{P}) \leqslant \max_{C \in \mathcal{C}} \mathrm{tw}(C, \mathcal{P}_C) < 2^h - 2.$$

This proves the claim. $\Diamond$

We can now finish the proof of the theorem. Let

$$c_{5.6}(h,d) = \max\{2d-3, c_{5.6.1}(h,d)\}$$

Let G be a graph, let $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G, such that G has no $\mathcal{F}$-rich model of $U_{h,d}$. If G is not connected, then we apply the induction hypothesis to every connected component C with the family $\mathcal{F}|_C$. This gives a set $S_C \subseteq V(C)$ and a partition $\mathcal{P}_C$ of S_C. Then, $S = \bigcup_C S_C$ and $\mathcal{P} = \bigcup_C \mathcal{P}_C$ are as desired. Now assume that G is connected. By Lemma 5.3, we assume that $\mathcal{D}$ is natural.

Recall that $U_{h,d}$ consists of d connected components isomorphic to $K_1 \oplus U_{h-1,d}$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of G such that H contains an $\mathcal{F}|_H$-rich model of $K_1 \oplus U_{h-1,d}$. Since G has no $\mathcal{F}$-rich model of $U_{h,d}$, there are no d pairwise disjoint members of $\mathcal{F}'$. Hence, by Lemma 1.17 and Lemma 5.4, there is a set $Z \subseteq V(G)$ which is the union of $2(d-1)-1 = 2d-3$ bags of $\mathcal{D}$ that intersects every member of $\mathcal{F}'$, and for every connected component C of $G-Z$, $N_G(V(C))$ intersects at most two connected components of $G-V(C)$. Observe that by construction, $G-Z$ has no $\mathcal{F}|_{G-Z}$-rich model of $K_1 \oplus U_{h-1,d}$. Hence, by Claim 5.6.2, there exists $S_0 \subseteq V(G)$ and a partition $\mathcal{P}_0$ of S_0 such that

5.6.2.(a) $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}|_{G-Z}$,

5.6.2.(b) for every connected component C of $G-(Z \cup S_0)$, $N_{G-Z}(V(C))$ intersects at most $2^{h+1}-4$ connected components of $G-V(C)$,

5.6.2.(c) $\mathcal{P}_0$ has $\mathcal{D}$-width at most $c_{5.6.1}(h,d)$, and

5.6.2.(d) $\mathrm{tw}(G-Z, \mathcal{P}_0) < 2^h - 2$.

Then, let

$$S = Z \cup S_0 \quad \text{and} \quad \mathcal{P} = \{Z\} \cup \mathcal{P}_0.$$

We now prove that S and $\mathcal{P}$ satisfy (A) to (D).

For every $F \in \mathcal{F}$, either $V(F) \cap Z \neq \emptyset$, or $F \in \mathcal{F}|_{G-Z}$, which implies $V(F) \cap S_0 \neq \emptyset$ by 5.6.2.(a). This proves (A).

Let C be a connected component of $G-S$. Let C' be the connected component of $G-Z$ containing C. Then $N_G(V(C'))$ intersects at most two connected components of $G-V(C')$ by the definition of Z, and $N_{G-Z}(V(C))$ intersects at most $2^{h+1}-4$ connected components of $G-(Z \cup V(C))$ by 5.6.2.(b). Hence $N_G(V(C))$ intesects at most $2 + 2^{h+1} - 4 = 2^{h+1} - 2$ connected components of $G - V(C)$. This proves (B).

Since $\mathcal{P}_0$ has $\mathcal{D}$-width at most $c_{5.6.1}(h,d)$, and because Z is included in the union of at most $2d-3$ bags of $\mathcal{D}$, we deduce that $\mathcal{P}$ has $\mathcal{D}$-width at most $c_{5.6}(h,d)$. This proves (C).

Finally, since $\mathcal{P} = \{Z\} \cup \mathcal{P}_0$,

$$\mathrm{tw}(G,\mathcal{P}) \leqslant 1 + \mathrm{tw}(G-Z, \mathcal{P}_0) < 1 + 2^h - 2 = 2^h - 1.$$

This proves (D) and concludes the proof of the theorem. □

Chapter 6

Centered colorings and weak coloring numbers

The results presented in this chapter are joint work with Jędrzej Hodor, Hoang La, and Piotr Micek, and are partially announced in [HLMR24a] and [HLMR25].

In this chapter, we prove generic bounds on three families of graph parameters, namely the centered chromatic numbers, weak coloring numbers, and fractional treedepth-fragility rates. The impatient reader interested only in centered colorings in K_t-minor-free graphs can read the preliminaries (Section 6.1) and then directly Section 6.9, up to Section 6.9.3. First, we recall the definitions of these families of parameters. See Section 1.3.3 for a proper introduction.

Let G be a graph, let q be a positive integer, and let C be a set of colors. A coloring $\varphi\colon V(G) \to C$ of G is *q-centered* if for every connected subgraph H of G, either φ uses more than q colors on $V(H)$, or there is a color that appears exactly once on $V(H)$. The *q-centered chromatic number* of G, denoted by $\operatorname{cen}_q(G)$, is the least nonnegative integer k such that G admits a q-centered coloring using k colors.

Let G be a graph, let $\Pi(G)$ be the set of all vertex orderings of G, let $\sigma \in \Pi(G)$, and let q be a nonnegative integer. For all u and v vertices of G, we say that v is *weakly q-reachable from u in (G,σ)*, if there exists a path P between u and v in G containing at most q edges such that for every $w \in V(P)$, $v \leqslant_\sigma w$. Let $\operatorname{WReach}_q[G,\sigma,u]$ be the set of vertices that are weakly q-reachable from u in (G,σ). The *q-th weak coloring number* of G is defined as

$$\operatorname{wcol}_q(G) = \min_{\sigma\in\Pi(G)} \max_{u\in V(G)} |\operatorname{WReach}_q[G,\sigma,u]|.$$

Let G be a graph and let q be a positive integer. The *q-th fractional treedepth-fragility rate* of G, denoted by $\operatorname{ftdfr}_q(G)$, is the minimum positive integer k such that there exists a family $\mathcal{Y}$ of subsets of $V(G)$ such that

(i) $\operatorname{td}(G-Y) \leqslant k$ for every $Y \in \mathcal{Y}$;

(ii) there exists a probability distribution λ on $\mathcal{Y}$ such that for every $u \in V(G)$, we have $\sum_{u\in Y\in\mathcal{Y}} \lambda(Y) \leqslant \frac{1}{q}$.

To determine the growth rates of these parameters in minor-closed classes of graphs, we introduce two graph parameters, the rooted 2-treedepth and simple rooted 2-treedepth. We now recall their definitions.

Let $\mathcal{X}$ be a class of graphs. We define $\mathbf{T}(\mathcal{X})$ as the class of all the graphs G such that there is a rooted forest decomposition $(F,(W_x \mid x \in V(F)))$ of G of adhesion at most 1 such that for every $x \in V(F)$, $G[W_x \setminus W_{\mathrm{p}(F,x)}] \in \mathcal{X}$ if x is not a root, and $|W_x| \leqslant 1$ if x is a root. We define, for every nonnegative integer t, the classes $\mathcal{R}_t$ and $\mathcal{S}_t$ by

$$\mathcal{R}_t = \begin{cases} \text{only the null graph} & \text{if } t=0,\\ \text{all edgeless graphs} & \text{if } t=1,\\ \text{all forests} & \text{if } t=2,\\ \mathbf{T}(\mathcal{R}_{t-1}) & \text{if } t\geqslant 3,\end{cases} \quad\text{and}\quad \mathcal{S}_t = \begin{cases} \text{only the null graph} & \text{if } t=0,\\ \text{all edgeless graphs} & \text{if } t=1,\\ \text{all linear forests} & \text{if } t=2,\\ \mathbf{T}(\mathcal{S}_{t-1}) & \text{if } t\geqslant 3.\end{cases}$$

We are ready to define the two key parameters, namely, rooted 2-treedepth (denoted by $\operatorname{rtd}_2(\cdot)$) and simple rooted 2-treedepth (denoted by $\operatorname{srtd}_2(\cdot)$). For every graph G, let

$$\operatorname{rtd}_2(G) = \min\{t \in \mathbb{N} \mid G \in \mathcal{R}_t\},$$
$$\operatorname{srtd}_2(G) = \min\{t \in \mathbb{N} \mid G \in \mathcal{S}_t\}.$$

In this chapter, we prove the following bounds on centered chromatic numbers, weak coloring numbers, and fractional treedepth-fragility rates in minor-closed classes of graphs.

Theorem 1.34. *For every integer t with $t \geqslant 2$, for every graph X, there exists an integer c such that, for every X-minor-free graph G, for every integer q with $q \geqslant 2$, if $\operatorname{srtd}_2(X) \leqslant t$, then*

$$\operatorname{cen}_q(G) \leqslant c \cdot q^{t-1},$$
$$\operatorname{cen}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2};$$

and if $\operatorname{rtd}_2(X) \leqslant t$ and $t \geqslant 3$, then

$$\operatorname{cen}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\operatorname{cen}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2} \log q.$$

Theorem 1.35. *For every integer t with $t \geqslant 2$, for every graph X, there exists an integer c such that, for every X-minor-free graph G, for every integer q with $q \geqslant 2$, if $\operatorname{srtd}_2(X) \leqslant t$, then*

$$\operatorname{wcol}_q(G) \leqslant c \cdot q^{t-1},$$
$$\operatorname{wcol}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2};$$

and if $\operatorname{rtd}_2(X) \leqslant t$, then

$$\operatorname{wcol}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\operatorname{wcol}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2} \log q.$$

Theorem 1.36. *For every integer t with $t \geqslant 2$, for every graph X, there exists an integer c such that, for every X-minor-free graph G, for every integer q with $q \geqslant 2$, if $\operatorname{srtd}_2(X) \leqslant t$, then*

$$\operatorname{ftdfr}_q(G) \leqslant c \cdot q^{t-1},$$
$$\operatorname{ftdfr}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2};$$

and if $\operatorname{rtd}_2(X) \leqslant t$, then

$$\operatorname{ftdfr}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\operatorname{ftdfr}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2} \log q.$$

The final proofs of these theorems are given in, respectively, Sections 6.7 to 6.9. These three theorems, together with known lower bounds (see Appendix A), yield the following classification of growth rates of centered chromatic numbers, weak coloring numbers, and fractional treedepth-fragility rates in minor-closed classes of graphs.

Corollary 1.37. *Let $\operatorname{par} \in \{\operatorname{cen}, \operatorname{wcol}, \operatorname{ftdfr}\}$, let $\mathcal{X}$ be a nonempty family of nonnull graphs, and let $\mathcal{C}$ be the class of all graphs G such that X is not a minor of G for all $X \in \mathcal{X}$. Let $s = \min\{\operatorname{srtd}_2(X) \mid X \in \mathcal{X}\}$ and $t = \min\{\operatorname{rtd}_2(X) \mid X \in \mathcal{X}\}$. For every integer q with $q \geqslant 2$,*

- *if $t \leqslant 1$ or $(s,t) = (2,2)$, then*

$$\max_{G \in \mathcal{C}} \operatorname{par}_q(G) = \Theta(1),$$

- *if $(s,t) = (2,3)$, then*

$$\max_{G \in \mathcal{C}} \operatorname{par}_q(G) = \begin{cases} \Theta(q) & \text{if } \operatorname{par} = \operatorname{cen}, \\ \Theta(\log q) & \text{if } \operatorname{par} \in \{\operatorname{wcol}, \operatorname{ftdfr}\}, \end{cases}$$

- *if* $3 \leqslant t = s$, *then for every integer* q *with* $q \geqslant 2$,

$$\Omega(q^{t-2}) \leqslant \max_{G \in \mathcal{C}} \mathrm{par}_q(G) \leqslant \mathcal{O}(q^{t-1}),$$

and moreover $\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \Theta(q^{t-2})$ *if* $\mathcal{X}$ *contains a planar graph,*

- *if* $3 \leqslant t = s - 1$, *then for every integer* q *with* $q \geqslant 2$,

$$\Omega(q^{t-2} \log q) \leqslant \max_{G \in \mathcal{C}} \mathrm{par}_q(G) \leqslant \mathcal{O}(q^{t-1} \log q),$$

and moreover $\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \Theta(q^{t-2} \log q)$ *if* $\mathcal{X}$ *contains a planar graph.*

Proof. An important observation is that $\max_{q \in \mathbb{N}_{>0}} \mathrm{par}_q(G) = \mathrm{td}(G)$. If $t \leqslant 1$ or $(s,t) = (2,2)$, then $\mathcal{X}$ contains a path, and so $\mathcal{C}$ has bounded treedepth by Proposition 1.12, and it follows that $\max_{G \in \mathcal{C}} \mathrm{par}_q(G)$ is bounded by a constant independent of q.

If $(t,s) = (2,3)$, then $\mathcal{X}$ contains a forest, and so by Theorem 1.10, $\mathcal{C}$ has bounded treewidth. First suppose $\mathrm{par} \in \{\mathrm{wcol}, \mathrm{ftdfr}\}$. Then, by Theorems 1.35 and 1.36, $\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(\log q)$. Moreover, since $s \geqslant 3$, every path is in $\mathcal{C}$. It is a simple fact that paths have at least logarithmic weak coloring numbers and fractional treedepth-fragility rates, see [JM22] and [DS20]. Now suppose $\mathrm{par} = \mathrm{cen}$. Then, by Theorem 1.34 for $t = 3$, $\max_{G \in \mathcal{C}} \mathrm{cen}_q(G) = \mathcal{O}(q)$. Again, since $\mathcal{C}$ contains every path, $\mathcal{C}$ has unbounded treedepth. Therefore $\max_{G \in \mathcal{C}} \mathrm{cen}_q(G)$ is not bounded by a constant independent of q, and so $\max_{G \in \mathcal{C}} \mathrm{cen}_q(G) \geqslant q + 1$.

If $3 \leqslant t = s$, then, by Theorems 1.34 to 1.36, $\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-1})$. If moreover $\mathcal{X}$ contains a planar graph, then $\mathcal{C}$ has bounded treewidth by Theorem 1.10, and so $\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-2})$. For the lower bound, by the definition of t and s, we have $\mathcal{R}_{t-1} \subseteq \mathcal{C}$. But by the lower bounds recalled in Appendix A.1, there are graphs in $\mathcal{R}_{t-1}$ with par_q in $\Omega(q^{t-2})$, which gives the claimed lower bound.

If $3 \leqslant t = s - 1$, then, by Theorems 1.34 to 1.36, $\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-1} \log q)$. If moreover $\mathcal{X}$ contains a planar graph, then $\mathcal{C}$ has bounded treewidth by Theorem 1.10, and so $\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-2} \log q)$. For the lower bound, by the definition of t and s, we have $\mathcal{S}_t \subseteq \mathcal{C}$. But by the lower bounds recalled in Appendix A.1, there are graphs in $\mathcal{S}_t$ with par_q in $\Omega(q^{t-2} \log q)$, which gives the claimed lower bound. □

6.1 Preliminaries

We denote by $\mathcal{E}$ the class of all the edgeless graphs.

An *ordering* of a finite set S is a sequence $\sigma = (x_1, \dots, x_{|S|})$ of all the elements of S. We write $\min_\sigma S = x_1$, $\max_\sigma S = x_{|S|}$, and $x_i \leqslant_\sigma x_j$ if and only if $i \leqslant j$, for every $i, j \in [|S|]$. A collection $\mathcal{P}$ of subsets of a nonempty set S is a *partition* of S if elements of $\mathcal{P}$ are nonempty, pairwise disjoint, and $\bigcup \mathcal{P} = S$.

Let F be a rooted forest. For a vertex $x \in V(F)$, we denote by F_x the subtree induced by all the descendants of x in F. An *elimination ordering* of a tree T rooted in $s \in V(T)$ is an ordering $(x_1, \dots, x_{|V(T)|})$ of $V(T)$ such that $x_1 = s$ and for every $i \in \{2, \dots, |V(T)|\}$, $N(x_i) \cap \{x_j \mid j \in [i-1]\} = \{\mathrm{p}(T, x_i)\}$.

Let G_1, G_2 be two graphs. We denote by $G_1 \sqcup G_2$ the disjoint union of G_1 and G_2, and by $G_1 \oplus G_2$ the graph obtained from $G_1 \sqcup G_2$ by adding all possible edges with one endpoint in

$V(G_1)$ and the other in $V(G_2)$. For every positive integer k, for every graph G, we write $k \cdot G$ for the union of k disjoint copies of G.

A *tree partition* of a graph G is a pair $(T, \mathcal{P})$, where T is a rooted tree and $\mathcal{P} = (P_x \mid x \in V(T))$ is a partition of $V(G)$ such that for every edge uv in G either there is $x \in V(T)$ with $u, v \in P_x$ or there is an edge xy in T with $u \in P_x$ and $v \in P_y$. A *tree partition* of (G, S), where G is a graph and $S \subseteq V(G)$ is a tree partition $(T, (P_x \mid x \in V(T)))$ of $G[S]$ such that for every connected component C of $G - S$, there exists an edge xy in T such that $N_G(V(C)) \subseteq P_x \cup P_y$. Recall that a *path partition* of G (resp. (G, S)) is a tree partition $(T, (P_x \mid x \in V(T)))$ of G (resp. (G, S)) where T is a path.

Let $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ be a tree decomposition of a graph G. An *elimination ordering* of $\mathcal{W}$ is an ordering $(u_1, \dots, u_{|V(G)|})$ of $V(G)$ such that for every $i \in [|V(G)|]$, there exists $x \in V(T)$ such that

$$\bigcup \{W_z \mid z \in V(T) \text{ and } u_i \in W_z\} \cap \{u_j \mid j \in [i-1]\} \subseteq W_x.$$ [1]

We conclude this section by introducing the notion of layered Robertson-Seymour decompositions. Recall that a *layered tree decomposition* of a graph G is a pair $(\mathcal{W}, \mathcal{L})$ where $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ is a tree decomposition of G, and $\mathcal{L} = (L_i \mid i \in \mathbb{N})$ is a *layering* of G. The *width* of $(\mathcal{W}, \mathcal{L})$ is $\max |W_x \cap L_i|$ over all $x \in V(T)$ and $i \in \mathbb{N}$.

For a graph G, a tree decomposition $\mathcal{W} = (T, (W_x \mid x \in V(T)))$, and $x \in V(T)$, the *torso* of W_x in G, and $\mathcal{W}$, denoted by $\mathrm{torso}_{G,\mathcal{W}}(W_x)$, is the graph with the vertex set W_x and where two distinct vertices u, v are adjacent if $uv \in E(G)$, or there exists $y \in N_T(x)$ such that $u, v \in W_x \cap W_y$ [2]. For a graph G and a positive integer c, a *layered Robertson-Seymour decomposition* of G (*layered RS-decomposition* for short) of width at most c is a tuple

$$(T, \mathcal{W}, \mathcal{A}, \mathcal{D}, \mathcal{L})$$

where T is a tree, and

(lrs1) $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ is a tree decomposition of G of adhesion at most c;

(lrs2) $\mathcal{A} = (A_x \mid x \in V(T))$ where $A_x \subseteq W_x$ and $|A_x| \leqslant c$ for every $x \in V(T)$;

(lrs3) $\mathcal{D} = (\mathcal{D}_x \mid x \in V(T))$ where $\mathcal{D}_x = (T_x, (D_{x,z} \mid z \in V(T_x)))$ is a tree decomposition of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$ for every $x \in V(T)$;

(lrs4) $\mathcal{L} = (\mathcal{L}_x \mid x \in V(T))$ where $\mathcal{L}_x = (L_{x,i} \mid i \in \mathbb{N})$ is a layering of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$ for every $x \in V(T)$; and

(lrs5) $|D_{x,z} \cap L_{x,i}| \leqslant c$ for all $x \in V(T)$, $z \in V(T_x)$, and $i \in \mathbb{N}$.

See Figure 6.1. Dujmović, Morin, and Wood proved in [DMW17] that, for every fixed positive integer t, K_t-minor-free graphs admit such decompositions with bounded width.

Theorem 6.1 ([DMW17, Theorem 22 and Lemma 26])**.** *For every positive integer t, there is a positive integer $c_{\mathrm{LRS}}(t)$ such that every K_t-minor-free graph admits a layered RS-decomposition of width at most $c_{\mathrm{LRS}}(t)$.*

[1] Equivalently, $(u_1, \dots, u_{|V(G)|})$ is an elimination ordering of $\mathcal{W}$ if and only if it is a perfect elimination ordering of the chordal graph obtained from G by adding all possible edges between vertices in a same bag W_x for every $x \in V(T)$.

[2] Note that this definition of torso coincides with the graph $\mathrm{torso}_G(W_x, \mathcal{B})$ defined in Chapter 2 for $\mathcal{B} = \{\bigcup_{s \in V(S)} W_s \mid S \text{ connected component of } T - x\}$.

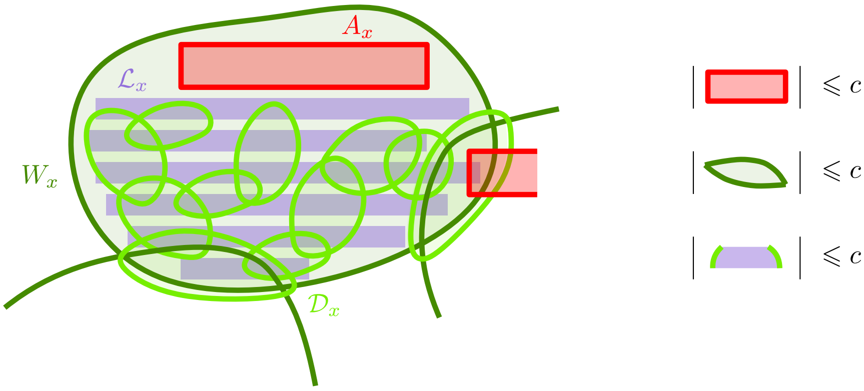


Figure 6.1: A bag W_x of $\mathcal{W}$ in a layered RS-decomposition $(\mathcal{T}, \mathcal{W}, \mathcal{A}, \mathcal{D}, \mathcal{L})$ of width at most c. The set A_x (in red) is included in W_x. The graph $\operatorname{torso}_{G,\mathcal{W}}(W_x) - A_x$ has a layering $\mathcal{L}_x$ (in purple), and a tree decomposition $\mathcal{D}_x$ (in green). Note that for every $y \in V(T) \setminus \{x\}$, $(W_x \cap W_y) \setminus A_x$ is a clique in $\operatorname{torso}_{G,\mathcal{W}}(W_x) - A_x$, and so, it is contained in a single bag of $\mathcal{D}_x$ and in at most two layers of $\mathcal{L}_x$.

6.2 The abstract framework

The proofs of our main theorems (Theorems 1.34 to 1.36) are by induction on (simple) rooted 2-treedepth. We present an abstract framework which will encapsulate the induction step for the three considered families of parameters (cen, wcol, and ftdfr).

6.2.1 Families of focused parameters

Let $\mathcal{Z}$ be the family of all pairs (G, S) where G is a graph and $S \subseteq V(G)$. We call a family

$$\operatorname{par} = (\operatorname{par}_q : \mathcal{Z} \to [0, \infty) \mid q \in \mathbb{N}_{>0})$$

a *family of focused parameters*[3].

The following definition encapsulates the properties that a family of focused parameters has to satisfy so that the abstract induction step can be performed.

We say that a family of focused parameters is *nice* if there exist positive integers b and b' such that for every positive integer q and every $(G, S) \in \mathcal{Z}$, we have

(z1) $\operatorname{par}_q(G, S) \leqslant \operatorname{par}_q(C, S \cap V(C))$ for some connected component C of G;

(z2) $\operatorname{par}_q(G, S) \leqslant |S|$,

(z3) $\operatorname{par}_q(G, S_1 \cup S_2) \leqslant \operatorname{par}_q(G, S_1) + \operatorname{par}_q(G - S_1, S_2)$ for all disjoint $S_1, S_2 \subseteq V(G)$;

(z4) for every tree partition $(T, (P_x \mid x \in V(T)))$ of (G, S),

$$\operatorname{par}_q(G, S) \leqslant b'(q+1) \cdot \max_{x \in V(T)} \operatorname{par}_{bq}(G_x, P_x)$$

where for every $x \in V(T)$, G_x is the subgraph of G induced by the union of $U_x = \bigcup\{P_z \mid z \in V(T_x)\}$ and all the vertex sets of the connected components of $G - S$ having a neighbor in U_x.

[3] In a slight abuse of notation, we write $\operatorname{par}_q(G, S)$ instead of $\operatorname{par}_q((G, S))$.

In Section 6.3, we define focused versions of $\mathrm{cen}_q(\cdot)$, $\mathrm{wcol}_q(\cdot)$, and $\mathrm{ftdfr}_q(\cdot)$; and we show that these family of focused parameters satisfy (z1)-(z4). Therefore, the abstract theorems, that we will state at the end of this section, can be applied to prove the induction step of our main theorems.

6.2.2 Coloring elimination property

To bound the growth rate of our families of focused parameters in minor-closed graph classes, we rely on the structure of the excluded minors – namely, rooted 2-treedepth and/or simple rooted 2-treedepth. The families with bounded rtd_2 and srtd_2 are built inductively in the same manner and they differ only in the base cases. Therefore, we extract the main property of these two parameters to characterize a family $\mathcal{X}$ of graphs for which the inductive step of our main theorems would apply.

Let $\mathcal{X}$ be a class of graphs. We say that $\mathcal{X}$ has the *coloring elimination property* if, for every positive integer k, for every $X \in \mathcal{X}$, there exists $Y \in \mathcal{X}$ such that for all sets $S_1, \dots, S_k \subseteq V(Y)$ such that $\bigcup_{i\in[k]} S_i = V(Y)$, there exists $i \in [k]$ such that there is an S_i-rooted model of X in Y. In this case, we say that Y *witnesses* the coloring elimination property of $\mathcal{X}$ for X and k.

In Lemma 6.21, in Section 6.4, we show that, for every positive integer t, $\mathcal{R}_t$ and $\mathcal{S}_t$ have the coloring elimination property.

6.2.3 The abstract induction step

The general idea of the abstract statement below is the following. Consider a nice family of focused parameters par_q. Let $\mathcal{X}$ be a class of graphs like $\mathcal{R}_t$ or $\mathcal{S}_t$. We would like to proceed by induction on the structure of excluded minors (on t) by showing that, if, for every fixed $X \in \mathcal{X}$, par_q is in $\mathcal{O}(g(q))$ in X-minor-free graphs for some function g, then par_q is bounded by $\mathcal{O}(q \cdot g(q))$ in X'-minor-free graphs, for every fixed $X' \in \mathbf{T}(\mathcal{X})$ (recall that $\mathcal{R}_{t+1} = \mathbf{T}(\mathcal{R}_t)$ and $\mathcal{S}_{t+1} = \mathbf{T}(\mathcal{S}_t)$). To this end, we reinforce our induction hypothesis by introducing the notion of $(\mathrm{par}, \mathcal{X})$-bounding functions.

Let par be a family of focused parameters and let $\mathcal{X}$ be a class of graphs. A function $g\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ is $(\mathrm{par}, \mathcal{X})$*-bounding* if for every positive integer k, there exists a nonnegative integer α such that for every graph $X \in \mathcal{X}$, there exists a nonnegative integer $\beta(X)$ such that for every positive integer q, for every K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if G has no $\mathcal{F}$-rich model of X, then there exists $S \subseteq V(G)$ such that

(g1) $S \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$;

(g2) for every connected component C of $G - S$, $N_G(V(C))$ intersects at most α connected components of $G - V(C)$;

(g3) $\mathrm{par}_q(G, S) \leqslant \beta(X) \cdot g(q)$.

In this case, when k is fixed, we say that α and $\beta(\cdot)$ *witness* g being $(\mathrm{par}, \mathcal{X})$-bounding. Essentially, we require that excluding an $\mathcal{F}$-rich model of a graph X in G implies the existence of a set $S \subseteq V(G)$ hitting $\mathcal{F}$ such that the parameter focused on S is bounded.

Now we can state the main abstract statement of this section.

Theorem 6.2. *Let* par *be a nice family of focused parameters, let* $\mathcal{X}$ *be a class of graphs having the coloring elimination property and closed under disjoint union and leaf addition. Let* $g\colon \mathbb{N}_{>0} \to$

$\mathbb{N}_{>0}$ *be a function. If* g *is* $(\mathrm{par}, \mathcal{X})$*-bounding, then there exists a positive integer* b *such that* $q \mapsto q \cdot g(bq)$ *is* $(\mathrm{par}, \mathbf{T}(\mathcal{X}))$*-bounding.*

In practice, we split this induction into the two following steps.

Theorem 6.3. *Let* par *be a nice family of focused parameters, let* $\mathcal{X}$ *be a class of graphs having the coloring elimination property, and let* $g\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$*. If* g *is* $(\mathrm{par}, \mathcal{X})$*-bounding, then there exists a positive integer* b *such that* $q \mapsto q \cdot g(bq)$ *is* $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$*-bounding.*

Theorem 6.4. *Let* par *be a nice family of focused parameters, let* $\mathcal{X}$ *be a nonempty class of graphs closed under disjoint union and leaf addition, and let* $g\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$*. If* g *is* $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$*-bounding, then* g *is* $(\mathrm{par}, \mathbf{T}(\mathcal{X}))$*-bounding.*

Clearly, Theorems 6.3 and 6.4 imply Theorem 6.2. We prove Theorems 6.3 and 6.4 in Section 6.5. To deduce Theorems 1.34 to 1.36, we will choose carefully the families $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ of focused parameters to consider, and show that the desired function (typically $q \mapsto q^{t-2} \log q$ or $q \mapsto q^{t-2}$) is $(\mathrm{par}, \mathcal{R}_t)$-bounding or $(\mathrm{par}, \mathcal{S}_t)$-bounding. To do so, we proceed by induction on t. The base cases are proved in Sections 6.6 to 6.9, while the induction is carried out by Theorems 6.3 and 6.4. The plans of these final proofs is depicted in Figures 6.2 and 6.3. We did not depict the plan for the part of Theorem 1.35 corresponding to graphs of unbounded treewidth, but the argument is very similar.

6.3 The considered nice families of focused parameters

In this section, we define the families of focused parameters we will consider in order to prove Theorems 1.34 to 1.36, and we prove that they are nice.

6.3.1 Weak coloring numbers

Let G be a graph, let q be a nonnegative integer, let $S \subseteq V(G)$, let σ be an ordering of S, let $u \in V(G)$, and let $v \in S$. We say that v *is weakly* q*-reachable from* u *in* (G, S, σ) if there is a (u,v)-path P in G of length at most q such that $\min_\sigma(V(P) \cap S) = v$. We denote by $\mathrm{WReach}_q[G, S, \sigma, u]$ the set of all the weakly q-reachable vertices from u in (G, S, σ) and we write $\mathrm{wcol}_q(G, S, \sigma) = \max_{u \in V(G)} |\mathrm{WReach}_q[G, S, \sigma, u]|$. Finally, let $\mathrm{wcol}_q(G, S)$ be the minimum value of $\mathrm{wcol}_q(G, S, \sigma)$ among all orderings σ of S. For each of the defined objects, we drop S when $S = V(G)$. Namely, v is weakly q-reachable from u in (G, σ) whenever v is weakly q-reachable from u in $(G, V(G), \sigma)$, $\mathrm{WReach}_q[G, \sigma, u] = \mathrm{WReach}_q[G, V(G), \sigma, u]$, $\mathrm{wcol}_q(G, \sigma) = \mathrm{wcol}_q(G, V(G), \sigma)$, and $\mathrm{wcol}_q(G) = \mathrm{wcol}_q(G, V(G))$. This matches the definition given in Chapter 1. See an illustration in Figure 6.4.

Next, we state some simple observations concerning weak coloring numbers. Ultimately, we will show that $(\mathrm{wcol}_q \mid q \in \mathbb{N}_{>0})$ is a nice family of focused parameters.

Observation 6.5. *Let* G *be a graph, let* $S \subseteq V(G)$*, and let* $\mathcal{C}$ *be the family of the connected components of* G*. For every positive integer* q*, we have*

$$\mathrm{wcol}_q(G, S) = \max_{C \in \mathcal{C}} \mathrm{wcol}_q(C, S \cap V(C)).$$

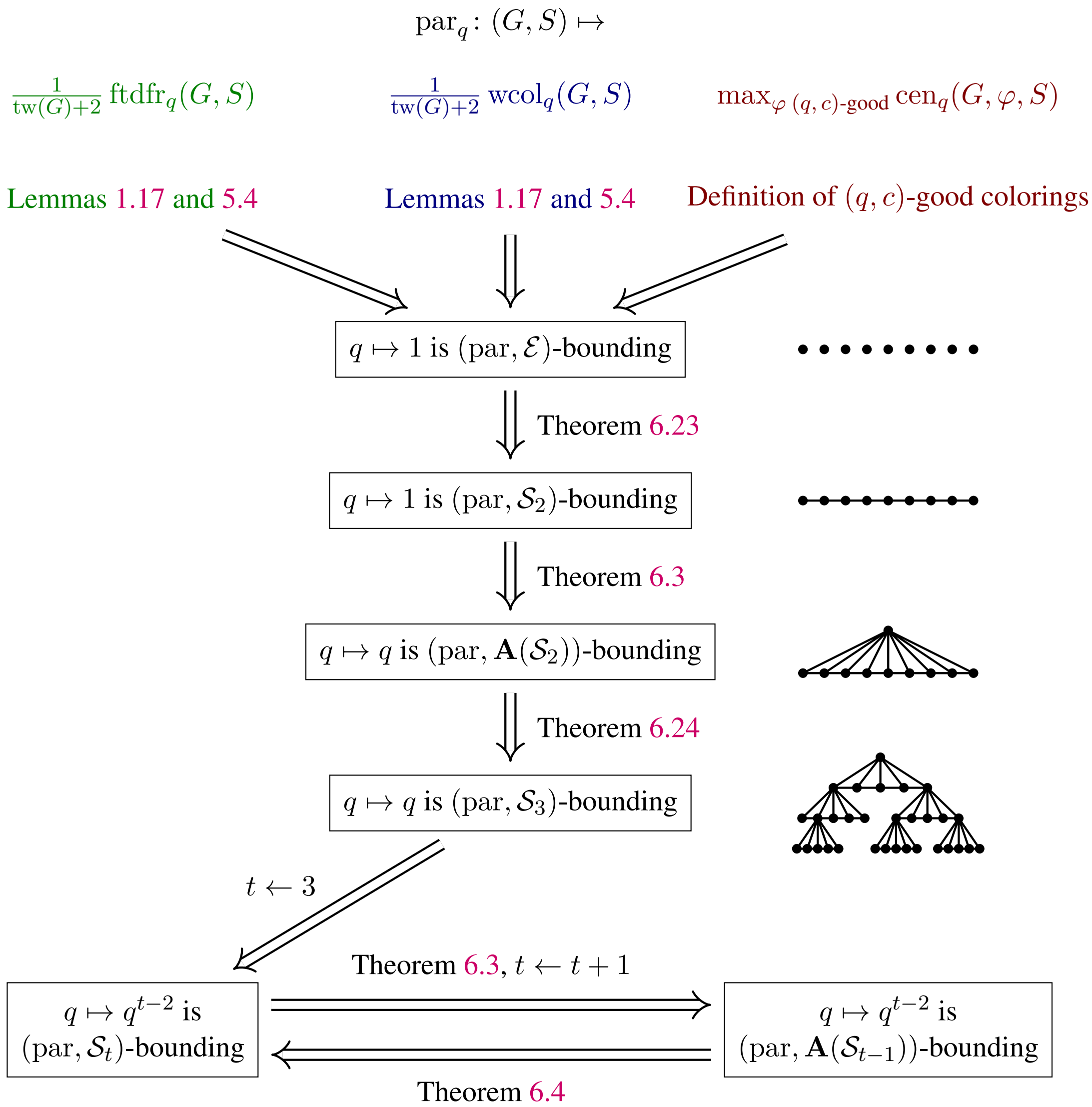


Figure 6.2: Road map for the proofs of Theorems 1.34 to 1.36 for the part corresponding to excluding a graph in $\mathcal{S}_t$. We show that, for a carefully chosen nice family of focused parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$, $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding for every integer t with $t \geqslant 3$. The main part of the induction consists in applying Theorem 6.3 and Theorem 6.4. Since Theorem 6.4 can be applied to $\mathcal{X} = \mathcal{S}_{t-1}$ only if $t \geqslant 4$ (because $\mathcal{X}$ should be closed under leaf addition), the cases $t = 2$ and $t = 3$ are proved separately. On the right hand is depicted a typical excluded rich minor for each step of the base case.

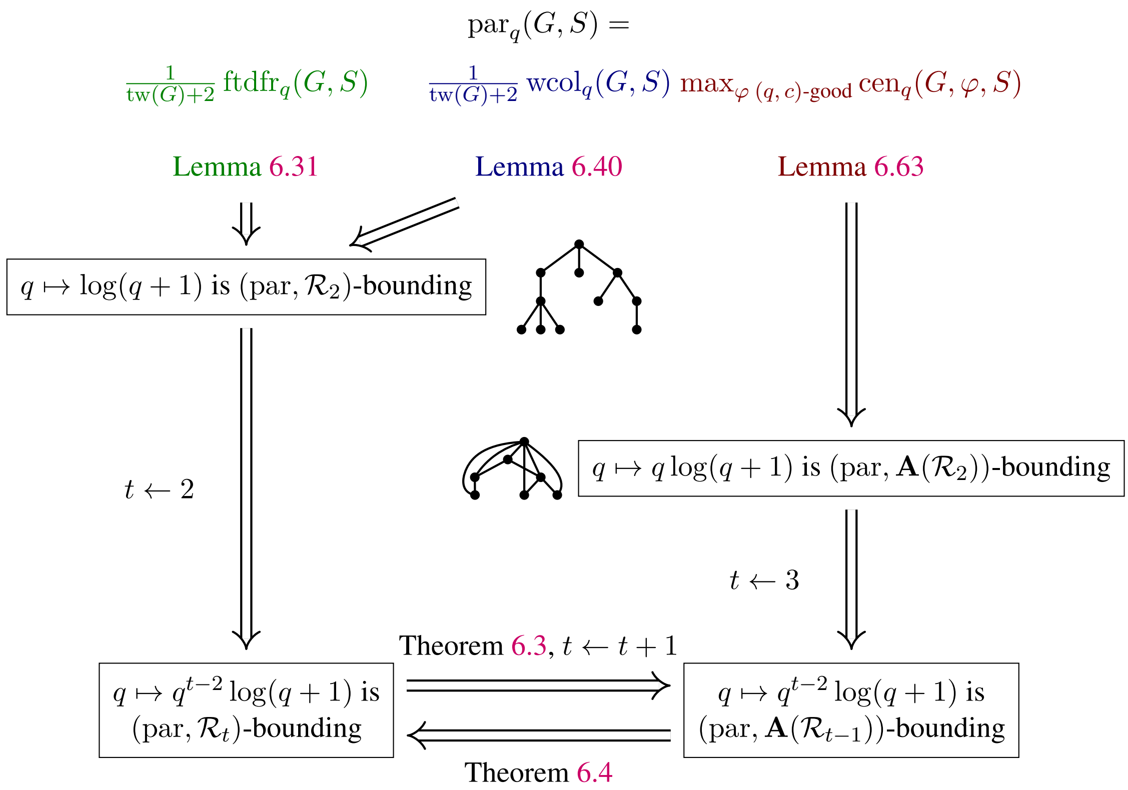


Figure 6.3: Road map for the proofs of Theorems 1.34 to 1.36 for the part corresponding to excluding a graph in $\mathcal{R}_t$. The base case corresponds to excluding a graph in $\mathcal{R}_2$, that is a forest.

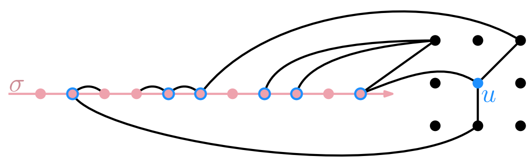


Figure 6.4: The pink vertices correspond to the set S. The vertices in S highlighted blue are in $\mathrm{WReach}_3[G, S, \sigma, u]$.

Observation 6.6. *Let G be a graph and let S_1, S_2 be disjoint subsets of $V(G)$. For every positive integer q, we have*

$$\mathrm{wcol}_q(G, S_1 \cup S_2) \leqslant \mathrm{wcol}_q(G, S_1) + \mathrm{wcol}_q(G - S_1, S_2).$$

Observation 6.5 is clear from the definition and to see Observation 6.6, it suffices to order all the vertices of S_1 before all the vertices of S_2.

To derive an upper bound on weak coloring numbers of trees, it suffices to root a given tree and take an elimination ordering of the vertices. In such an ordering, for every positive integer q, a vertex weakly q-reaches only its $q + 1$ closest ancestors (including itself). See Figure 6.5.

Observation 6.7. *Let T be a rooted tree. Let σ be an elimination ordering of T. Then*

$$\mathrm{wcol}_q(T, \sigma) \leqslant q + 1.$$

We extend this idea to tree partitions of graphs.

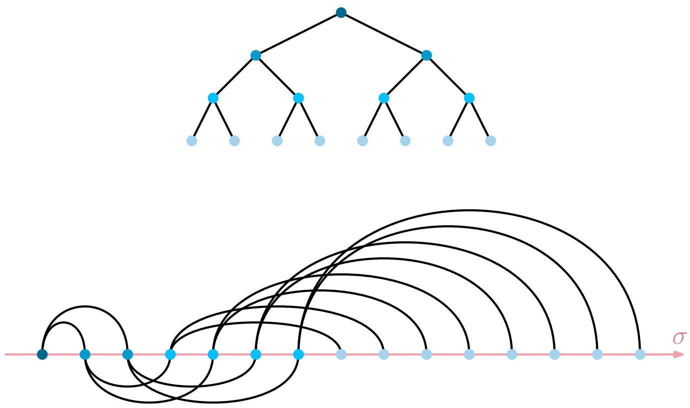


Figure 6.5: An example of an elimination ordering of a complete binary tree of height 3.

Lemma 6.8. *Let G be a graph, let $S \subseteq V(G)$, let $(T, (P_x \mid x \in V(T)))$ be a tree partition of (G, S). For every $x \in V(T)$, let G_x be the subgraph of G induced by the union of $U_x = \bigcup\{P_z \mid z \in V(T_x)\}$ and all the vertex sets of the connected components of $G - S$ having a neighbor in U_x. For every positive integer q, we have*

$$\mathrm{wcol}_q(G, S) \leqslant (q + 1) \cdot \max_{x \in V(T)} \mathrm{wcol}_q(G_x, P_x).$$

Proof. Let q be a positive integer and let $k = \max_{x \in V(T)} \mathrm{wcol}_q(G_x, P_x)$. Let $(x_1, \dots, x_{|V(T)|})$ be an elimination ordering of T. For every $x \in V(T)$, let σ_x witness $\mathrm{wcol}_q(G_x, P_x) \leqslant k$. Let σ be an ordering of $S = \bigcup_{x \in V(T)} P_x$ obtained by concatenating $\sigma_{x_1}, \dots, \sigma_{x_{|V(T)|}}$.

It suffices to show that $\mathrm{wcol}_q(G, S, \sigma) \leqslant (q + 1)k$. Let $u \in V(G)$ and consider $W = \mathrm{WReach}_q[G, S, \sigma, u]$. Assume that $W \neq \emptyset$. If $u \in S$, let $x(u) \in V(T)$ such that $u \in P_{x(u)}$. Otherwise, let C be the connected component of u in $G - S$ and let $x(u)$ be the vertex of T furthest from $\mathrm{root}(T)$ such that $N_G(V(C))$ intersects $P_{x(u)}$. Observe that for every $x \in V(T)$, $W \cap P_x \subseteq \mathrm{WReach}_q[G_x, P_x, \sigma_x, u]$. Moreover, if x is not among the closest $q + 1$ ancestors of $x(u)$ (including $x(u)$ itself), then $W \cap P_x = \emptyset$. Altogether, this implies that $|W| \leqslant (q + 1)k$ as desired. $\square$

Lemma 6.9. *The family* $(\mathrm{wcol}_q \mid q \in \mathbb{N}_{>0})$ *is nice.*

Proof. Observation 6.5 implies (z1), (z2) is clear from the definition, Observation 6.6 implies (z3), and (z4) with $b' = b = 1$ follows from Lemma 6.8. □

In Theorem 1.35, we give variants of each upper bound by trading off a q factor for $\mathrm{tw}(G)+1$. In order to obtain the bound depending on treewidth, we apply our techniques to the following family of focused parameters.

Lemma 6.10. *For every graph G, for every $S \subseteq V(G)$, and for every positive integer q, let*

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\,\mathrm{wcol}_q(G,S)$$

The family of focused parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ *is nice.*

Proof. Since $\mathrm{tw}(G) \geqslant -1$ for every graph G, the parameters are well-defined. The family is nice by monotonicity of treewidth under taking subgraphs and because the family $(\mathrm{wcol}_q \mid q \in \mathbb{N}_{>0})$ is nice by Lemma 6.9. □

6.3.2 Centered colorings

Let S be a set. A *coloring* of S is a function $\varphi\colon S \to C$ for some set C. For each $u \in S$, we say that $\varphi(u)$ is the *color* of u and subsequently we say that $\varphi(S)$ is the *set of colors used by* φ. For two colorings φ_1 and φ_2 of S, we define the coloring $\varphi_1 \times \varphi_2$ of S, called the *product coloring* of φ_1 and φ_2, by $(\varphi_1 \times \varphi_2)(u) = (\varphi_1(u), \varphi_2(u))$ for every $u \in S$. Given $S' \subseteq S$, an element $u \in S'$ is a φ*-center* of S' if the color of u is unique in S' under φ, in other words, $\varphi(u) \notin \varphi(S' \setminus \{u\})$. Let G be a graph. A *coloring* of G is a coloring of $V(G)$. When H is a subgraph of G and φ is a coloring of G, we denote by $\varphi|_{V(H)}$, the restriction of φ to $V(H)$. Recall that a coloring φ of G is a q-centered coloring of G for a positive integer q if for every connected subgraph H of G, either $|\varphi(V(H))| > q$ or $V(H)$ has a φ-center.

In the case of centered colorings, we consider a focused family of parameters defined on precolored graphs.

Let q be a positive integer, let G be a graph, and let φ be a coloring of G. For every $S \subseteq V(G)$, we define $\mathrm{cen}_q(G,\varphi,S)$ to be the least nonnegative integer k such that there exists a coloring $\psi\colon S \to C$ for some cardinality k set C, such that for every connected subgraph H of G such that $V(H)$ intersects S, one of the following is true:

(i) $|\varphi(V(H))| > q$, or

(ii) $|(\varphi \times \psi)(V(H) \cap S)| > q$, or

(iii) there is a $(\varphi \times \psi)$-center of $V(H) \cap S$.

A crucial observation is that, if $S = V(G)$, and if ψ is such a coloring, then $\varphi \times \psi$ is a q-centered coloring of G. We begin with a few straightforward observations on $\mathrm{cen}_q(\cdot,\cdot,\cdot)$.

Observation 6.11. *Let G be a graph, let φ be a coloring of G, let $S \subseteq V(G)$, and let $\mathcal{C}$ be the family of all the connected components of G. For every positive integer q, we have*

$$\mathrm{cen}_q(G,\varphi,S) = \max_{C \in \mathcal{C}} \mathrm{cen}_q(C, \varphi|_{V(C)}, S \cap V(C)).$$

Observation 6.12. *Let G be a graph, let φ be a coloring of G, and let S_1, S_2 be disjoint subsets of $V(G)$. For every positive integer q, we have*

$$\mathrm{cen}_q(G, \varphi, S_1 \cup S_2) \leqslant \mathrm{cen}_q(G, \varphi, S_1) + \mathrm{cen}_q(G - S_1, \varphi|_{V(G)\setminus S_1}, S_2).$$

Observation 6.13. *Let G be a graph, let φ be a coloring of G, and let $S \subseteq V(G)$.*

$$\mathrm{cen}_q(G, \varphi, S) \leqslant |S|.$$

Observation 6.11 follows just from the definition of $\mathrm{cen}_q(G, \varphi, S)$, i.e., each connected subgraph H of G is a subgraph of a connected component of G. On the other hand, to see Observation 6.12, we use two disjoint sets of $\mathrm{cen}_q(G, \varphi, S_1)$ and $\mathrm{cen}_q(G - S_1, \varphi|_{V(G)\setminus S_1}, S_2)$ colors to color respectively S_1 and S_2. Finally, to see Observation 6.13 it suffices to take for ψ an injective coloring of S.

We will use very particular precolorings, namely (q, c)-good colorings (see Section 6.9.2.1). However, in order to obtain a nice family of focused parameters we only require the following property.

For every graph G, let Φ_G be a set of colorings of G. We say that the family $\Phi = \{\Phi_G \mid G \text{ is a graph}\}$ is *hereditary* if for all graphs G and H such that H is a subgraph of G, for all $\varphi \in \Phi_G$, we have $\varphi|_{V(H)} \in \Phi_H$.

Lemma 6.14. *Let $\Phi = (\Phi_H \mid H$ is a graph$)$ be a hereditary family of sets of graph colorings. For every graph G, for every $S \subseteq V(G)$, and for every positive integer q, let*

$$\mathrm{par}_q(G, S) = \max\{\mathrm{cen}_q(G, \varphi, S) \mid \varphi \in \Phi_G\}.$$

The family $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ is nice.

Proof. Let q be a positive integer, let G be a graph, let $S \subseteq V(G)$, and let $\varphi \in \Phi_G$.

Since Φ is hereditary, (z1) follows from Observation 6.11 and (z3) follows from Observation 6.12. Additionally, (z2) follows from Observation 6.13.

Let $(T, (P_x \mid x \in V(T)))$ be a tree partition of (G, S) with T rooted in $r \in V(T)$. For every $x \in V(T)$, let G_x be the subgraph of G induced by the union of $U_x = \bigcup\{P_z \mid z \in V(T_x)\}$ and all the vertex sets of the connected components of $G - S$ having a neighbor in U_x. Let $\varphi \in \Phi_G$ and let $k = \max\{\mathrm{cen}_q(G_x, \varphi|_{V(G_x)}, P_x) \mid x \in V(T)\}$. For every $x \in V(T)$, let $\psi_x \colon P_x \to [k]$ witness $\mathrm{cen}_q(G_x, \varphi|_{V(G_x)}, P_x) \leqslant k$. Let $\psi \colon S \to [k] \times \{0, \dots, q\}$ be defined by

$$\psi(u) = (\psi_x(u), \mathrm{dist}_T(r, x) \bmod (q+1))$$

for every $x \in V(T)$ and for every $u \in P_x$.

We claim that ψ witnesses $\mathrm{cen}_q(G, \varphi, S) \leqslant (q+1)k$. Consider a connected subgraph H of G intersecting S. Let x be the vertex of T closest to the root such that $V(H) \cap P_x \neq \emptyset$. Observe that $H \subseteq G_x$. Since ψ_x witnesses $\mathrm{cen}_q(G_x, \varphi|_{V(G_x)}, P_x) \leqslant k$, one of the following is true: $|\varphi(V(H))| > q$ or $|(\varphi \times \psi_x)(V(H) \cap P_x)| > q$ or there is a $(\varphi \times \psi_x)$-center u of $V(H) \cap P_x$. If one of the first two cases holds, the claim holds. Thus, assume that the latter holds. If u is a $(\varphi \times \psi)$-center of $V(H) \cap S$, then again the claim holds. Otherwise, there exists $y \in V(T)$ distinct from x such that

$$\mathrm{dist}_T(r, x) \equiv \mathrm{dist}_T(r, y) \bmod (q+1),$$

and $V(H) \cap P_y \neq \emptyset$. Since H is connected, x is an ancestor of y by the choice of x. Altogether, this implies that $\mathrm{dist}_T(x, y) > q$ and $V(H)$ intersects P_z for every z in the (x, y)-path in T. This implies that $|(\varphi \times \psi)(V(H) \cap S)| \geqslant |\psi(V(H) \cap S)| > q$. Therefore, ψ witnesses $\mathrm{cen}_q(G, \varphi, S) \leqslant (q+1)k$. Since φ was an arbitrary coloring in Φ_G, we obtain (z4) with $b' = b = 1$. This concludes the proof of the lemma. □

6.3.3 Fractional treedepth-fragility rates

In Chapter 3, we defined the focused graph parameter $\mathrm{td}(\cdot, \cdot)$. An equivalent definition is the following. For every graph G and $S \subseteq V(G)$, if $\mathcal{C}$ is the family of the connected components of G, then

$$\mathrm{td}(G, S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \min_{u \in S} \mathrm{td}(G, S - \{u\}) + 1 & \text{if } G \text{ is connected, and} \\ \max_{C \in \mathcal{C}} \mathrm{td}(C, S \cap V(C)) & \text{if } V(G) \neq \emptyset \text{ and } G \text{ is not connected.} \end{cases}$$

Straight from the definition, we have the following observation.

Observation 6.15. *Let G be a graph, let $S \subseteq V(G)$, and let $\mathcal{C}$ be the family of all the connected components of G. We have*

$$\mathrm{td}(G, S) = \max_{C \in \mathcal{C}} \mathrm{td}(C, S \cap V(C)).$$

Lemma 6.16. *Let G be graph and let S_1, S_2 be disjoint subsets of $V(G)$. We have*

$$\mathrm{td}(G, S_1 \cup S_2) \leqslant \mathrm{td}(G, S_1) + \mathrm{td}(G - S_1, S_2).$$

Proof. We proceed by induction on $|V(G)|$. If $S_1 = \emptyset$, then the result is clear. Thus, assume $|S_1| \geqslant 1$. If G is not connected, then for every connected component C of G, by the induction hypothesis, $\mathrm{td}(C, (S_1 \cup S_2) \cap V(C)) \leqslant \mathrm{td}(C, S_1 \cap V(C)) + \mathrm{td}(C - (V(C) \cap S_1), S_2 \cap V(C)) \leqslant \mathrm{td}(G, S_1) + \mathrm{td}(G - S_1, S_2)$. This implies the result by Observation 6.15. If G is connected, then by the definition of $\mathrm{td}(G, S_1)$, there exists $u \in S_1$ such that $\mathrm{td}(G, S_1) = 1 + \mathrm{td}(G - u, S_1 \setminus \{u\})$. By the induction hypothesis, we have $\mathrm{td}(G - u, (S_1 \setminus \{u\}) \cup S_2) \leqslant \mathrm{td}(G - u, S_1 \setminus \{u\}) + \mathrm{td}(G - S_1, S_2) = \mathrm{td}(G, S_1) - 1 + \mathrm{td}(G - S_1, S_2)$. Since $\mathrm{td}(G, S_1 \cup S_2) \leqslant 1 + \mathrm{td}(G - u, (S_1 \setminus \{u\}) \cup S_2)$, this implies the result. □

Lemma 6.17. *Let G be a graph, let $S \subseteq V(G)$, and let $(T, (P_x \mid x \in V(T)))$ be a tree partition of (G, S). We have*

$$\mathrm{td}(G, S) \leqslant h \cdot \max_{x \in V(T)} \mathrm{td}(G_x, P_x)$$

where h is the vertex-height of T and for every $x \in V(T)$, G_x is the subgraph of G induced by the union of $U_x = \bigcup\{P_z \mid z \in V(T_x)\}$ and all the vertex sets of the connected components of $G - S$ having a neighbor in U_x.

Proof. The statement is clear when $S = \emptyset$, hence, assume otherwise. We proceed by induction on h. The statement is clear again when $h = 1$, hence, assume $h \geqslant 2$. By possibly removing the vertex sets of all the connected components of G disjoint from S, we suppose that every connected component of G intersects S, and so $G = G_r$. Let $k = \max_{x \in V(T)} \mathrm{td}(G_x, P_x)$ and let r be the root of T. For every child x of r, $(T_x, (P_z \mid z \in V(T_x)))$ is a tree partition of $(G_x, S \cap V(G_x))$. Since T_x has vertex-height at most $h-1$, by induction, $\mathrm{td}(G_x, S \cap V(G_x)) \leqslant (h-1)\max_{z \in V(T_x)} \mathrm{td}(G_z, P_z) \leqslant (h-1)k$. By Observation 6.15, $\mathrm{td}(G - P_r, S - P_r) \leqslant \max\{\mathrm{td}(G_x, S \cap V(G_x)) \mid x \in V(T-r), \mathrm{p}(T,x) = r\} \leqslant (h-1)k$. By Lemma 6.16, $\mathrm{td}(G, S) \leqslant \mathrm{td}(G_r, P_r) + \mathrm{td}(G - P_r, S - P_r) \leqslant k + (h-1)k = h \cdot k$ since $G_r = G$. This concludes the proof. □

Let q be a positive integer, G be a graph, and $S \subseteq V(G)$. The *q-th fractional treedepth-fragility rate* of (G, S), denoted by $\mathrm{ftdfr}_q(G, S)$, is the minimum nonnegative integer k such that there exists a family $\mathcal{Y}$ of subsets of S such that

(i) $\mathrm{td}(G - Y, S \setminus Y) \leqslant k$ for every $Y \in \mathcal{Y}$, and

(ii) there exists a probability distribution λ on $\mathcal{Y}$ such that for every $u \in S$, we have $\sum_{u \in Y \in \mathcal{Y}} \lambda(Y) \leqslant \frac{1}{q}$.

We say that a probability distribution satisfying (ii) is *q-thin*. In other words, the probability distribution of a random variable Y taking subsets of S as values is q-thin if and only if $\Pr[u \in \mathsf{Y}] \leqslant \frac{1}{q}$ for every $u \in S$. We will use the following abbreviation. For a positive integer k, writing that "a random variable Y witnesses $\mathrm{ftdfr}(G, S) \leqslant k$", we mean that Y takes values in the set $\{Y \subseteq S \mid \mathrm{td}(G - Y, S \setminus Y) \leqslant k\}$ and the probability distribution of Y is q-thin. Furthermore, slightly abusing notation, we sometimes write that "Y satisfies some statement" where Y is a random variable, by which we mean that the statement is satisfied for every element that can be a value of Y.

Lemma 6.18. *For every graph G, for every $S \subseteq V(G)$, and for every positive integer q, let*

$$\mathrm{par}_q(G, S) = \frac{1}{\mathrm{tw}(G) + 2} \mathrm{ftdfr}_q(G, S).$$

The family of focused parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ is nice.

Proof. First, observe that, for every graph G, $1/(\mathrm{tw}(G) + 2)$ is well-defined since $\mathrm{tw}(G) \geqslant -1$. Let G be a graph, let $S \subseteq V(G)$, and let q be a positive integer.

Let $\mathcal{C}$ be the family of the connected components of G. Let $k = \max_{C \in \mathcal{C}} \mathrm{ftdfr}_q(C, S \cap V(C))$. For every $C \in \mathcal{C}$, let Y_C witness $\mathrm{ftdfr}_q(C, S \cap V(C)) \leqslant k$. Let $\mathsf{Y} = \bigcup_{C \in \mathcal{C}} \mathsf{Y}_C$. Using Observation 6.15, it is easy to verify that $\mathrm{td}(G - \mathsf{Y}, S - \mathsf{Y}) \leqslant k$. Let $u \in S$ and let $C \in \mathcal{C}$ such that $u \in V(C)$. Since the probability distribution of Y_C is q-thin, we have $\Pr[u \in \mathsf{Y}] = \Pr[u \in \mathsf{Y}_C] \leqslant \frac{1}{q}$. Thus, Y witnesses $\mathrm{ftdfr}_q(G, S) \leqslant k$. By definition of k, there is $C \in \mathcal{C}$ such that $\mathrm{ftdfr}_q(C, S \cap V(C)) = k$. Therefore, by monotonicity of treewidth, $\mathrm{par}_q(G, S) \leqslant \mathrm{par}_q(C, S \cap V(C))$, and so, (z1) holds.

To show (z2), consider a random variable Y that is always equal to the empty set. Then, since $\mathrm{td}(G, S) \leqslant |S|$, and because for every $u \in S$ we have $\Pr[u \in \mathsf{Y}] = 0 \leqslant \frac{1}{q}$, we deduce that Y witnesses $\mathrm{ftdfr}_q(G, S) \leqslant |S|$, which gives (z2).

Let S_1, S_2 be disjoints subsets of S whose union is S. By monotonicity of treewidth, in order to obtain (z3), it suffices to prove that $\mathrm{ftdfr}_q(G, S_1 \cup S_2) \leqslant \mathrm{ftdfr}_1(G, S_1) + \mathrm{ftdfr}_q(G - S_1, S_2)$. Let Y_1 and Y_2 witness $\mathrm{ftdfr}_q(G, S_1) = k_1$ and $\mathrm{ftdfr}_q(G - S_1, S_2) = k_2$ respectively. Let $\mathsf{Y} = \mathsf{Y}_1 \cup \mathsf{Y}_2$. By Lemma 6.16,

$$\mathrm{td}(G - \mathsf{Y}, S - \mathsf{Y}) \leqslant \mathrm{td}(G - \mathsf{Y}_1, S_1 - \mathsf{Y}_1) + \mathrm{td}((G - S_1) - \mathsf{Y}_2, S_2 - \mathsf{Y}_2) \leqslant k_1 + k_2.$$

Let $u \in S$ and assume that $u \in S_i$ for some $i \in [2]$. Since the probability distribution of Y_i is q-thin, we have $\Pr[u \in \mathsf{Y}] = \Pr[u \in \mathsf{Y}_i] \leqslant \frac{1}{q}$. It follows that Y witnesses $\operatorname{ftdfr}_q(G,S) \leqslant k_1 + k_2$, which completes the proof of (z3).

To show (z4), consider a tree partition $(T,(P_x \mid x \in V(T)))$ of (G,S) with T rooted in $r \in V(T)$. For every $x \in V(T)$, let G_x be the subgraph of G induced by the union of $U_x = \bigcup\{P_z \mid z \in V(T_x)\}$ and all the vertex sets of the connected components of $G - S$ having a neighbor in U_x. Let $k = \max_{x \in V(T)} \operatorname{ftdfr}_{2q}(G_x, P_x)$. For every $x \in V(T)$, let Y_x witness $\operatorname{ftdfr}_{2q}(G_x, P_x) \leqslant k$.

For each $i \in [2q]$, let $L_i = \{x \in V(T) \mid \operatorname{dist}_T(r,x) \equiv i \bmod 2q\}$. Let Y' be a random variable with the uniform distribution on the set $\left\{\bigcup_{x \in L_i} P_x \mid i \in [2q]\right\}$, and let $\mathsf{Y} = \mathsf{Y}' \cup \bigcup_{x \in V(T)} \mathsf{Y}_x$. For every $x \in V(T)$, let $\mathsf{P}_x = P_x - \mathsf{Y}$. Let F be the forest obtained from T by removing every vertex $x \in V(T)$ such that $\mathsf{P}_x = \emptyset$. We root each connected component of F in its vertex which is the closest to r in T. Let C be a connected component of $G - \mathsf{Y}$, and let T_C be the subtree of F such that $V(\mathsf{C}) \cap \bigcup_{x \in V(\mathsf{T}_\mathsf{C})} \mathsf{P}_x \neq \emptyset$. Observe that $(\mathsf{T}_\mathsf{C}, (\mathsf{P}_x \mid x \in V(\mathsf{T}_\mathsf{C})))$ is a tree partition of $(\mathsf{C}, (S - \mathsf{Y}) \cap V(\mathsf{C}))$. Since Y' contains all parts of vertices in T of the same depth every $2q$ layers, all trees in F have vertex-height at most $2q-1$. Therefore, by Lemma 6.17,

$$\begin{aligned}\operatorname{td}(\mathsf{C}, (S - \mathsf{Y}) \cap V(\mathsf{C})) &\leqslant (2q-1) \max_{x \in V(\mathsf{T}_\mathsf{C})} \operatorname{td}(\mathsf{G}_x, \mathsf{P}_x)\\ &\leqslant (2q-1) \max_{x \in V(T)} \operatorname{td}(G_x - \mathsf{Y}_x, P_x - \mathsf{Y}_x) \leqslant (2q-1)k,\end{aligned}$$

where for each $x \in V(T)$, G_x is the subgraph of C induced by the union of $\mathsf{U}_x = \bigcup\{\mathsf{P}_z \mid z \in V((\mathsf{T}_\mathsf{C})_x)\}$ and all the vertex sets of the connected components of $\mathsf{C} - (S \setminus \mathsf{Y})$ having a neighbor in U_x.

Finally, let $u \in S$ and let $x \in V(T)$ such that $u \in P_x$. By the definition of Y' and because the probability distribution of Y_x is q-thin, the union bound yields

$$\Pr[u \in \mathsf{Y}] = \Pr[u \in \mathsf{Y}'] + \Pr[u \in \mathsf{Y}_x] \leqslant \frac{1}{2q} + \frac{1}{2q} = \frac{1}{q}.$$

Therefore, (z4) holds with $b' = b = 2$. This concludes the proof of the lemma. □

6.4 Coloring elimination property of $\mathcal{R}_t$ and $\mathcal{S}_t$

In this section, we show that $\mathcal{R}_t$ and $\mathcal{S}_t$ have the coloring elimination property for every nonnegative integer t. First, we show that the operators $\mathbf{A}$ and $\mathbf{T}$ preserve the property under some mild assumptions (see Lemma 6.20). Therefore, to conclude, it suffices to check the base cases. In general, we work with classes of graphs that are closed under taking minors. However, this assumption is not necessary, as shown in the next lemma.

Lemma 6.19. *Let $\mathcal{X}$ be a class of graphs. Let $\mathcal{X}' = \{X' \mid X'$ is a minor of X for some $X \in \mathcal{X}\}$. The class $\mathcal{X}$ has the coloring elimination property if and only if $\mathcal{X}'$ has the coloring elimination property.*

Proof. Suppose that $\mathcal{X}'$ has the coloring elimination property. Let k be a positive integer and let $X \in \mathcal{X}$. Let $Y' \in \mathcal{X}'$ witness the coloring elimination property of $\mathcal{X}'$ for X and k (since $\mathcal{X} \subseteq \mathcal{X}'$). By the definition of $\mathcal{X}'$, there is $Y \in \mathcal{X}$ such that Y contains a model $(B'_y \mid y \in V(Y'))$ of Y'. Fix

$S_1, \ldots, S_k \subseteq V(Y)$ whose union is $V(Y)$. For every $i \in [k]$, let $S'_i = \{y \in V(Y') \mid B'_y \cap S_i \neq \emptyset\}$. Since $\bigcup_{i\in[k]} S_i = V(Y)$, we have $\bigcup_{i\in[k]} S'_i = V(Y')$. Fix $j \in [k]$ and an S'_j-rooted model $(B_x \mid x \in V(X))$ of X in Y'. For every $x \in V(X)$, let $D_x = \bigcup_{y\in B_x} B'_y$. By construction, $(D_x \mid x \in V(X))$ is an S_j-rooted model of X in Y. Therefore, $\mathcal{X}$ has the coloring elimination property.

Next, suppose that $\mathcal{X}$ has the coloring elimination property. Let k be a positive integer and let $X' \in \mathcal{X}'$. By the definition of $\mathcal{X}'$, there is $X \in \mathcal{X}$ such that X contains a model $(B'_x \mid x \in V(X'))$ of X'. Let $Y \in \mathcal{X}$ witness the coloring elimination property of $\mathcal{X}$ for X and k. Observe that $Y \in \mathcal{X}'$. Fix $S_1, \ldots, S_k \subseteq V(Y)$ whose union is $V(Y)$. Fix $j \in [k]$ and an S_j-rooted model $(B_x \mid x \in V(X))$ of X in Y. Finally, $(\bigcup_{y\in B'_x} B_y \mid x \in V(X'))$ is an S_j-rooted model of X' in Y, which shows that $\mathcal{X}'$ has the coloring elimination property. □

Lemma 6.20. *Let $\mathcal{X}$ be a class of graphs closed under disjoint union. If $\mathcal{X}$ has the coloring elimination property, then $\mathbf{A}(\mathcal{X})$ and $\mathbf{T}(\mathcal{X})$ have the coloring elimination property.*

Proof. By Lemma 6.19, we can assume that $\mathcal{X}$ is closed under taking minors. The statement is clear for when $\mathcal{X}$ consists only of the null graph, thus, assume that $\mathcal{X}$ contains a nonnull graph. Let k be a positive integer.

For every $X \in \mathbf{T}(\mathcal{X})$, we say that a vertex $u \in V(X)$ is a *root* of X if there exists a rooted forest decomposition $(F, (W_a \mid a \in V(F)))$ of X of adhesion at most 1 such that for every $a \in V(F)$ that is not a root, $G[W_a - W_{\mathrm{p}(F,a)}] \in \mathcal{X}$; for every a that is a root, $|W_a| \leqslant 1$; and there exists a root r of F such that $W_r = \{u\}$. For every $X \in \mathbf{T}(\mathcal{X})$, X has a root u unless X is the null graph because $\mathcal{X}$ is closed under taking minors (in particular, for every $X \in \mathcal{X}$ and $u \in V(X)$, we have $X - u \in \mathcal{X}$). This definition also applies to the members of $\mathbf{A}(\mathcal{X})$ since $\mathbf{A}(\mathcal{X}) \subseteq \mathbf{T}(\mathcal{X})$. For every $X \in \mathbf{A}(\mathcal{X})$, for every root u of X, $X - u \in \mathcal{X}$.

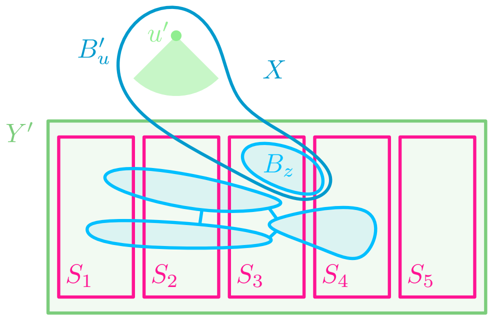


Figure 6.6: The vertex u' corresponds to K_1 in $K_1 \oplus Y'$. The union of the sets $S_1, \ldots, S_5$ is $V(Y')$. In the figure there is an S_3-rooted model of $K_1 \sqcup (X - u)$ in Y' (light-blue). We add u' to the branch set B_z where z corresponds to K_1 in $K_1 \sqcup (X - u)$, obtaining an S_3-rooted model of X in Y.

Claim 6.20.1. *Let X be a nonnull graph in $\mathbf{A}(\mathcal{X})$ and let u be a root of X. There exists $Y \in \mathbf{A}(\mathcal{X})$ with a root u' such that for every family of sets $S_1, \ldots, S_k$ whose union is $V(Y)$, Y contains an S_i-rooted model $(B_x \mid x \in V(X))$ of X for some $i \in [k]$ such that $u' \in B_u$.*

Proof of the claim. The proof is illustrated in Figure 6.6. Let $X' = K_1 \sqcup (X - u)$. Since $\mathcal{X}$ contains a nonnull graph and is closed under taking minors, it contains K_1. Since u is a root of X

and $X \in \mathbf{A}(\mathcal{X})$, $X - u \in \mathcal{X}$. In particular, $X' \in \mathcal{X}$ as $\mathcal{X}$ is closed under disjoint union. We denote by z the vertex corresponding to K_1 in X'. Let $Y' \in \mathcal{X}$ witness the coloring elimination property in $\mathcal{X}$ for X' and k. Furthermore, let $Y = K_1 \oplus Y'$, and let u' be the vertex of Y corresponding to K_1. In particular, $Y \in \mathbf{A}(\mathcal{X})$ and u' is a root of Y.

Consider a family of sets $S_1, \dots, S_k \subseteq V(Y)$ whose union is $V(Y) = V(Y') \cup \{u'\}$. There exists $i \in [k]$ such that Y' contains an $(S_i - \{u'\})$-rooted model $(B'_x \mid x \in V(X'))$ of X'. Let $B_u = B'_z \cup \{u'\}$ and $B_x = B'_x$ for every $x \in V(X') \setminus \{z\}$. We obtain that $(B_x \mid x \in V(X))$ is an S_i-rooted model of X in Y. Since $u' \in B_u$, this completes the proof of the claim. $\Diamond$

Claim 6.20.1 implies that $\mathbf{A}(\mathcal{X})$ has the coloring elimination property. A stronger assertion concerning the roots is needed in the proof of the next claim.

Claim 6.20.2. *Let $X \in \mathbf{T}(\mathcal{X})$ be connected and let u be a root of X. There exists $Y \in \mathbf{T}(\mathcal{X})$ and a root u' of Y, such that for every family of sets $S_1, \dots, S_k$ whose union is $V(Y)$, there is an S_i-rooted model $(B_x \mid x \in V(X))$ of X in Y for some $i \in [k]$ such that $u' \in B_u$.*

Proof of the claim. The objects defined in the proof of this claim are depicted in Figure 6.7. There exists a rooted forest decomposition $(T, (W_a \mid a \in V(T)))$ of X witnessing the fact that $X \in \mathbf{T}(\mathcal{X})$ and with $\{u\}$ as a root bag. Without loss of generality, $W_a \neq \emptyset$ for every $a \in V(T)$. Since X is connected, this implies that T is a tree. Fix such a rooted forest decomposition. Let r be the unique root of T. We proceed by induction on the vertex-height of T. When $T = K_1$, $X = K_1$ and the statement is clear. Now, suppose that $T \neq K_1$ and that the result holds for trees with smaller vertex-heights.

Let T_0 be the subtree of T induced by $\{a \in V(T) \mid u \in W_a\}$. Observe that for every $a \in V(T_0)$, $X[W_a \setminus \{u\}] \in \mathcal{X}$. Since $\mathcal{X}$ is closed under disjoint union, this implies that $X\left[\bigcup_{a \in V(T_0)} (W_a \setminus \{u\})\right] \in \mathcal{X}$. Therefore, $X_0 = X\left[\bigcup_{a \in V(T_0)} W_a\right]$ is in $\mathbf{A}(\mathcal{X})$ and u is a root of X_0. By Claim 6.20.1, there exists $Y_0 \in \mathbf{A}(\mathcal{X})$ and a root u' of Y_0 such that for every family of sets $S_1, \dots, S_k \subseteq V(Y_0)$ whose union is $V(Y_0)$, Y_0 contains an S_i-rooted model of X_0 for some $i \in [k]$, such that the branch set corresponding to u contains u'.

Let X_1 be the graph obtained from X by identifying all the vertices in $V(X_0)$ into a single vertex u_1. Observe that $X_1 \in \mathcal{X}$ since $\mathcal{X}$ is closed under taking minors. Let T_1 be the tree obtained from T by identifying the vertices in $V(T_0)$ into a single vertex r_1. For every $a \in V(T_1)$, let

$$W_{1,a} = \begin{cases} \{u_1\} & \text{if } a = r_1, \\ W_a & \text{otherwise.} \end{cases}$$

The rooted tree decomposition $(T_1, (W_{1,a} \mid a \in V(T_1)))$ of X_1 witnesses the fact that $X_1 \in \mathbf{T}(\mathcal{X})$. We claim that the vertex-height of T_1 is smaller than the vertex-height of T. Indeed, for every neighbor s or r in T, since X is connected and $W_s \neq \emptyset$, we have $u \in W_s$, and so $s \in V(T_0)$. This proves that $N_T(r) \subseteq V(T_0)$, and so T_1 has smaller vertex-height than T. Therefore, by the induction hypothesis, fix a graph $Y_1 \in \mathbf{T}(\mathcal{X})$ and a root u'_1 of Y_1 such that for every family of sets $S_1, \dots, S_k \subseteq V(Y_1)$ whose union is $V(Y_1)$, there is an S_i-rooted model of X_1 in Y_1 such that the branch set corresponding to u_1 contains u'_1. Since X_1 is connected, we assume without loss of generality that Y_1 is also connected.

For every $v \in V(Y_0)$, let $Y_{1,v}$ be a copy of Y_1. We label vertices in $Y_{1,v}$ in the following way. The vertex corresponding to u'_1 is labeled as v and a vertex corresponding to each $y \in V(Y_1) \setminus \{u'_1\}$

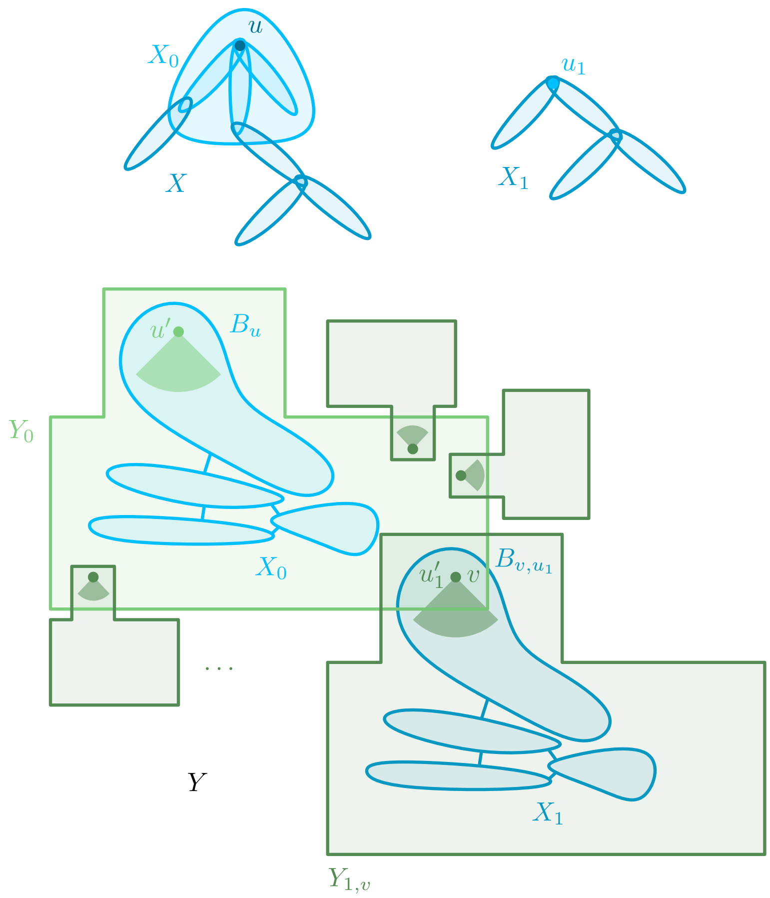


Figure 6.7: The graph X is split into two graphs according to its rooted tree decomposition witnessing $X \in \mathbf{T}(\mathcal{X})$: $X_0 \in \mathbf{A}(\mathcal{X})$ induced by all vertices of the bags containing the root u of X and $X_1 \in \mathbf{T}(\mathcal{X})$ obtained by contracting X_0. Since $X_0 \in \mathbf{A}(\mathcal{X})$, Claim 6.20.1 gives $Y_0 \in \mathbf{A}(\mathcal{X})$ which witness the coloring elimination property in $\mathbf{A}(\mathcal{X})$ for X_0 such that the root u' of Y_0 is in the branch set B_u. Since X_1 has a rooted tree decomposition witnessing $X_1 \in \mathbf{T}(\mathcal{X})$ with smaller vertex-height, the induction hypothesis gives a graph $Y_1 \in \mathbf{T}(\mathcal{X})$ witnessing the coloring elimination property in $\mathbf{T}(\mathcal{X})$ for X_1 and k, and such that the root u'_1 of Y_1 is in the branch set corresponding to u_1. We build a graph $Y \in \mathbf{T}(\mathcal{X})$ that witnesses the coloring elimination property in $\mathbf{T}(\mathcal{X})$ for X and k by taking, for every $v \in V(Y_0)$, a copy $Y_{1,v}$ of Y_1, and identifying v and the vertex corresponding to u'_1 in $Y_{1,v}$. This indeed yields a graph in $\mathbf{T}(\mathcal{X})$. Moreover, for any family of k sets covering $V(Y)$, a rooted model of X can be found using a rooted model of X_0 in Y_0 and rooted models of X_1 in the copies $Y_{1,v}$ for each $v \in V(Y_0)$.

is labeled as (y,v). Finally, let Y be obtained from the disjoint union of Y_0 and $Y_{1,v}$ for each $v \in V(Y_0)$ by identifying vertices with the same labels. We claim that Y witnesses the assertion of the claim.

First, we argue that $Y \in \mathbf{T}(\mathcal{X})$ and u' is a root of Y. For each $v \in V(Y_0)$, let $(T'_{1,v}, (W'_{1,v,a} \mid a \in V(T'_{1,v})))$ be a rooted tree decomposition of $Y_{1,v}$ witnessing $Y_1 \in \mathbf{T}(\mathcal{X})$, and with $T'_{1,v}$ rooted in $r'_{1,v}$ such that $W'_{1,r'_{1,v}} = \{(u'_1, v)\}$. Let T' be a tree obtained from a disjoint union of $T'_{1,v}$ for each $v \in V(Y_0)$ by identifying the roots of all copies into a vertex r'', and then adding the root r' with a single neighbor r''. We define $U_{r'} = \{u'\}$, $U_{r''} = V(Y_0)$, and $U_a = W'_{1,v,a}$ for every $a \in V(T') \setminus \{r', r''\}$ where $a \in V(T'_{1,v})$. Since $Y_0 \in \mathbf{A}(\mathcal{X})$, $Y_0 - u' \in \mathcal{X}$. Therefore, $(T', (U_a \mid a \in V(T')))$ is a rooted tree decomposition of Y witnessing that $Y \in \mathbf{T}(\mathcal{X})$. Additionally, $U_{r'} = \{u'\}$, hence, u' is a root of Y.

Let $S_1, \dots, S_k$ be a family of sets whose union is $V(Y)$. Let $v \in V(Y_0)$. Since $Y_{1,v}$ is a copy of Y_1 and v is a root of $Y_{1,v}$, there exists $i(v) \in [k]$ and an $(S_{i(v)} \cap V(Y_{1,v}))$-rooted model $\mathcal{M}_v = (B_{v,x} \mid x \in V(X_1))$ of X_1 in $Y_{1,v}$ with $v \in B_{v,u_1}$.

For each $j \in [k]$, let $S'_j = \{v \in V(Y_0) \mid i(v) = j\}$. Clearly, the union of $S'_1, \dots, S'_k$ is equal to $V(Y_0)$. Therefore, fix $\ell \in [k]$ and an S'_ℓ-rooted model $\mathcal{M}_0 = (B_x \mid x \in V(X_0))$ of X_0 in Y_0 with $u' \in B_u$ since u is a root of X_0 and u' is a root of Y_0. For every $x \in V(X_0)$, let

$$D_x = B_x \cup \bigcup_{v \in B_x} B_{v,u_1}.$$

In other words, we add to each branch set B in $\mathcal{M}_0$, the branch set of $\mathcal{M}_v$ containing v for every $v \in B$. Let $\mathcal{D} = (D_x \mid x \in V(X_0))$. Consider

$$\mathcal{M} = \mathcal{D} \cup \bigcup_{v \in S'_\ell} (\mathcal{M}_v \setminus \{B_{v,u_1}\}).$$

Observe that $\mathcal{M}$ is an S_ℓ-rooted model in Y. Moreover, $\mathcal{M}$ is a model of a supergraph of X in Y. Finally, recall that $u' \in V(Y_0)$ is a root of Y and $u' \in B_u \subseteq D_u$. This completes the proof of the claim. $\lozenge$

Since for every $X \in \mathbf{T}(\mathcal{X})$, there exists $X' \in \mathbf{T}(\mathcal{X})$ connected such that X is a subgraph of X', Claim 6.20.2 implies that $\mathbf{T}(\mathcal{X})$ has the coloring elimination property, which concludes the proof of the lemma. $\square$

Lemma 6.21. *For every nonnegative integer t, the classes of graphs $\mathcal{R}_t$ and $\mathcal{S}_t$ have the coloring elimination property.*

Proof. By Lemma 6.20, it is enough to show that $\mathcal{R}_0$, $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$ have the coloring elimination property.

Since $\mathcal{R}_0$ and $\mathcal{S}_0$ consist only of the null graph, the classes $\mathcal{R}_0$ and $\mathcal{S}_0$ have the coloring elimination property. Let k be a positive integer.

Consider $X \in \mathcal{S}_1$. Then X consists of $|V(X)|$ isolated vertices. Let Y consists of $k(|V(X)|-1)+1$ isolated vertices. For every $S_1, \dots, S_k$ whose union is $V(Y)$, there exists $i \in [k]$ such that $|S_i| \geqslant |V(X)|$ by the pigeonhole principle, and it follows that Y contains an S_i-rooted model of X. This proves that $\mathcal{S}_1$ has the coloring elimination property.

Consider $X \in \mathcal{S}_2$. Without loss of generality, X is connected, and so, X is a path, say on ℓ vertices. We set Y to be a path on $k(\ell-1)+1$ vertices. Let $S_1, \dots, S_k$ be sets whose union is

$V(Y)$. By the pigeonhole principle, there exists $i \in [k]$ such that $|S_i| \geqslant \ell$, and it follows that there is an S_i-rooted model of X in Y. This shows that $\mathcal{S}_2$ has the coloring elimination property and ends the proof. □

6.5 Proof of the induction step

Let par be a nice family of focused parameters, let $g\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$, and let $\mathcal{X}$ be a class of graphs. In several cases, to show that g is $(\mathrm{par}, \mathcal{X})$-bounding, we first show a weaker property that in fact implies the desired one, as shown in the lemma below.

The function g is *weakly* $(\mathrm{par}, \mathcal{X})$*-bounding* if for every positive integer k, for every graph $X \in \mathcal{X}$, there exist nonnegative integers $\alpha(X)$ and $\beta(X)$ such that for every positive integer q, for every K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if G has no $\mathcal{F}$-rich model of X, then there exists $S \subseteq V(G)$ such that

(gw1) $S \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$;

(gw2) for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $\alpha(X)$ connected components of $G - V(C)$;

(gw3) $\mathrm{par}_q(G, S) \leqslant \beta(X) \cdot g(q)$.

In this case, when k is fixed, we say that $\alpha(\cdot)$ and $\beta(\cdot)$ *witness* g being weakly $(\mathrm{par}, \mathcal{X})$-bounding.

Recall that $\mathcal{E}$ denotes the class of all the edgeless graphs.

Lemma 6.22. *Let* par *be a nice family of focused parameters, let $\mathcal{X}$ be a class of graphs, and let $g\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$. If g is $(\mathrm{par}, \mathcal{E})$-bounding and weakly $(\mathrm{par}, \mathcal{X})$-bounding, then g is $(\mathrm{par}, \mathcal{X})$-bounding.*

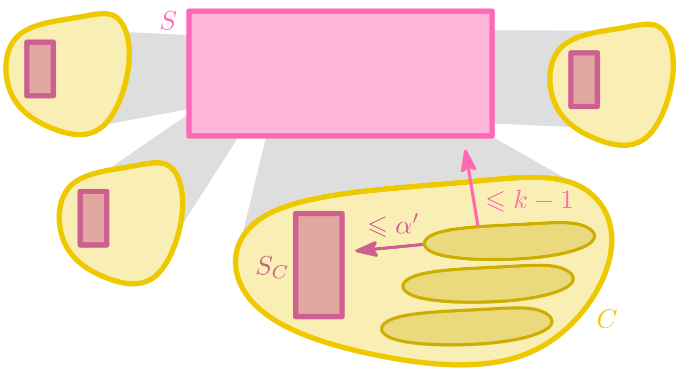


Figure 6.8: An illustration of the proof of Lemma 6.22.

Proof. Let k be a positive integer. Fix $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ witnessing that g is weakly $(\mathrm{par}, \mathcal{X})$-bounding and fix α' and $\beta'(\cdot)$ witnessing that g is $(\mathrm{par}, \mathcal{E})$-bounding. Since the only property distinguishing graphs in $\mathcal{E}$ is the number of vertices, we treat $\beta(\cdot)$ as a function taking nonnegative integers by setting $\beta'(d) = \beta'(\overline{K_d})$[4] for every positive integer d. Let

$$\alpha = \alpha' + (k-1) \quad \text{and} \quad \beta(X) = \beta_0(X) + \beta'\left(\binom{\alpha_0(X)}{k}(k-1) + 1\right) \text{ for every } X \in \mathcal{X}.$$

[4]For a positive integer n, we denote by $\overline{K_n}$ the edgeless graph on n vertices.

We show that α and $\beta(\cdot)$ witness g being $(\mathrm{par}, \mathcal{X})$-bounding. Let $X \in \mathcal{X}$, let q be a positive integer, let G be a K_k-minor-free graph, let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of X in G. There exists $S_0 \subseteq V(G)$ such that

(gw1') $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}$,

(gw2') for every connected component C of $G - S_0$, $N_G(V(C))$ intersects at most $\alpha_0(X)$ connected components of $G - V(C)$, and

(gw3') $\mathrm{par}_q(G, S_0) \leqslant \beta_0(X) \cdot g(q)$.

Let $\mathcal{C}$ be the family of all the connected components of $G - S_0$, and let $C \in \mathcal{C}$. Let $\mathcal{U}$ be the family of the vertex sets of all the connected components of $G - V(C)$ intersecting $N_G(V(C))$. By (gw2'), we have $|\mathcal{U}| \leqslant \alpha_0(X)$. Let $\mathcal{F}_C$ be the family of all the connected subgraphs H of C such that $|\{U \in \mathcal{U} \mid N_G(V(H)) \cap U \neq \emptyset\}| \geqslant k$. We claim that there no $N = \binom{\alpha_0(X)}{k}(k-1)+1$ pairwise disjoint members of $\mathcal{F}_C$. Otherwise, by the pigeonhole principle, there exists $U_1, \dots, U_k \in \mathcal{U}$ and $H_1, \dots, H_k \in \mathcal{F}'$ pairwise disjoint such that there is an edge between U_i and $V(H_j)$ in G for all $i, j \in [k]$. However, then $(U_i \cup V(H_i) \mid i \in [k])$ is a model of K_k in G, which is a contradiction proving that indeed there are no N pairwise disjoint members of $\mathcal{F}_C$. Therefore, there exists $S_C \subseteq V(G - S_0)$ such that

(g1') $V(F) \cap S_C \neq \emptyset$ for every $F \in \mathcal{F}_C$,

(g2') for every connected component C of $C - S_C$, $N_G(V(C))$ intersects at most α' connected components of $G - V(C)$, and

(g3') $\mathrm{par}_q(C, S_C) \leqslant \beta'(d) \cdot g(q)$.

We set

$$S = S_0 \cup \bigcup_{C \in \mathcal{C}} S_C.$$

See Figure 6.8. It suffices to verify that (g1), (g2), and (g3) hold. Since $S_0 \subseteq S$, by (gw1'), $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so, (g1) holds. For every connected component C' of $G - S$, there is a connected component $C \in \mathcal{C}$ such that $C' \subseteq C$. Since $C' \notin \mathcal{F}_C$ (by (g1')), $N_G(V(C'))$ intersects at most $k-1$ connected components of $G - V(C)$. Moreover, $N_G(V(C'))$ intersects at most α' connected components of $C - S_C$ (by (g2')). Therefore, $N_G(V(C'))$ intersects at most $\alpha' + (k-1) = \alpha$ connected components of $G - V(C')$. Finally,

$$\begin{aligned}
\mathrm{par}_q(G, S) &\leqslant \mathrm{par}_q(G, S_0) + \mathrm{par}_q\left(G - S_0, \textstyle\bigcup_{C \in \mathcal{C}} S_C\right) && \text{by (z3)}\\
&\leqslant \mathrm{par}_q(G, S_0) + \max_{C \in \mathcal{C}} \mathrm{par}_q(C, S_C) && \text{by (z1)}\\
&\leqslant \beta_0(X) \cdot g(q) + \beta'(N) \cdot g(q) && \text{by (gw3') and (g3')}\\
&= \beta(X) \cdot g(q).
\end{aligned}$$

We obtain that g is $(\mathrm{par}, \mathcal{X})$-bounding, which concludes the proof of the lemma. □

Proof of Theorem 6.3. Without loss of generality, assume that $\mathcal{X}$ contains a nonnull graph. Assume that g is $(\mathrm{par}, \mathcal{X})$-bounding. Let b and b' be as in (z4) for par. Let k be a positive integer. Fix α' and $\beta'(\cdot)$ witnessing that q is $(\mathrm{par}, \mathcal{X})$-bounding. Let $X \in \mathbf{A}(\mathcal{X})$ and $z \in V(X)$ such that $X - z \in \mathcal{X}$. Recall that $\mathcal{X}$ is closed under disjoint union and contains a nonnull graph, hence, $X' = K_1 \sqcup (X - z)$ is in $\mathcal{X}$. Let X'' witness the coloring elimination property of $\mathcal{X}$ for X' and α'.

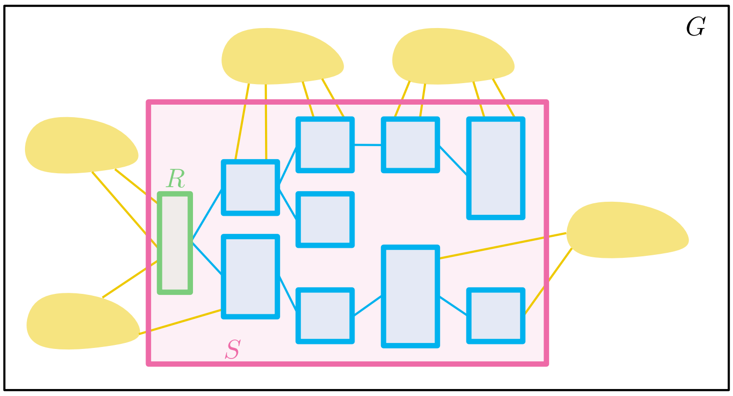


Figure 6.9: An illustration of the statement of Claim 6.3.1.

Claim 6.3.1. *Let G be a connected K_k-minor-free graph, let R be a nonempty set of at most α' vertices in G, let $\mathcal{F}$ be a family of connected subgraphs of G, and let q be a positive integer. If G has no $\mathcal{F}$-rich model of X, then there exist $S \subseteq V(G)$, a tree T rooted in $s \in V(T)$, and a tree partition $(T, (P_x \mid x \in V(T)))$ of (G, S) with $P_s = R$ such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;*

(b) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $2\alpha'$ connected components of $G - V(C)$;*

(c) *for every $x \in V(T)$,*

$$\mathrm{par}_q(G_x, P_x) \leqslant \max\{\beta'(X''), \alpha'\} \cdot g(q)$$

where for every $x \in V(T)$, G_x is the union of $U_x = \bigcup_{z \in V(T_x)} P_z$ and all the vertex sets of the connected components C of $G - S$ having a neighbor in U_x.

Proof of the claim. The statement is illustrated in Figure 6.9, and some ideas of the proof are illustrated in Figure 6.10. We proceed by induction on $|V(G)|$. If $\mathcal{F}|_{G-R} = \emptyset$, then the statement holds for $S = R$ and $T = K_1$. Thus, we suppose that $\mathcal{F}|_{G-R} \neq \emptyset$, and in particular $|V(G) \setminus R| > 0$.

Let $\mathcal{F}'$ be the family of all the connected subgraphs H of $G - R$ such that $R \cap N_G(V(H)) \neq \emptyset$ and $F \subseteq H$ for some $F \in \mathcal{F}$. We argue that there is no $\mathcal{F}'$-rich model of X'' in $G - R$. Suppose to the contrary that there is an $\mathcal{F}'$-rich model $(B_x \mid x \in V(X''))$ of X'' in $G - U$. For each $u \in U$, let $S_u = \{x \in V(X'') \mid u \in N_G(B_x)\}$. Since the model is $\mathcal{F}'$-rich, $\bigcup_{u \in R} S_u = V(X'')$. Therefore, there exists $u \in R$ such that X'' contains an S_u-rooted model of X'. As a consequence, there is an $\mathcal{F}$-rich model $(D_x \mid x \in V(X'))$ of X' in $G - R$ such that every branch set D_x contains a neighbor of u in G. Recall that $X' = K_1 \sqcup (X - z)$. Say that y is the vertex of K_1 in X'. Let $D'_z = D_y \cup \{u\}$ and $D'_x = D_x$ for every $x \in V(X' \setminus \{y\})$. It follows that $(D'_x \mid x \in V(X))$ is an $\mathcal{F}$-rich model of X in G, which is a contradiction. In turn, there is no $\mathcal{F}'$-rich model of X'' in $G - R$.

By the definition of α' and $\beta'(\cdot)$ applied to X'', $G - R$ and $\mathcal{F}'$, there exists a set $S_0 \subseteq V(G - R)$ such that (g1') $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}'$;

(g2') for every connected component C of $(G-R)-S_0$, $N_{G-R}(V(C))$ intersects at most α' connected components of $(G-R)-V(C)$;

(g3') $\mathrm{par}_q(G-R,S) \leqslant \beta'(X'') \cdot g(q)$.

Note that since G is connected and $\mathcal{F}|_{G-R}$ is nonempty, there is a connected component of $G-R$ containing a member of $\mathcal{F}$, and so $\mathcal{F}'$ is nonempty. It follows by (g1') that $S_0 \neq \emptyset$.

Let $\mathcal{C}_1$ be the family of all the connected components of $(G-R)-S_0$ with no neighbors in U. Consider $C \in \mathcal{C}_1$. Let A_C be the connected component of $G-R$ containing C, and let G_C be the graph obtained from A_C by contracting each connected component of $A_C - V(C)$ into a single vertex. Let R_C be the set of all the vertices resulting from these contractions, that is $R_C = V(G_C) \setminus V(C)$. Note that R_C is nonempty since G is connected, and $|U_C| \leqslant \alpha'$ by (g2'). Moreover, since $R \neq \emptyset$, $|V(G_C)| < |V(G)|$. Note also that G_C is a minor of G, and so G_C has no $\mathcal{F}|_C$-rich model of X. By the induction hypothesis applied to G_C, R_C, and $\mathcal{F}|_C$, there exist $S_C \subseteq V(G_C)$, a tree T_C rooted in $s_C \in V(T_C)$, and a tree partition $(T_C, (P_{C,x} \mid x \in V(T_C)))$ of (G_C, S_C) with $P_{C,s_C} = R_C$ such that

(a') $V(F) \cap S_C \neq \emptyset$ for every $F \in \mathcal{F}|_C$;

(b') for every connected component C' of $G_C - S_C$, $N_G(V(C'))$ intersects at most $2\alpha'$ connected components of $G_C - V(C')$;

(c') for every $x \in V(T_C)$ let $T_{C,x}$ be the subtree of T_C rooted in x, then

$$\mathrm{par}_q(G_{C,x}, P_{C,x}) \leqslant \max\{\beta'(X''), \alpha'\} \cdot g(q)$$

where for every $x \in V(T_C)$, $G_{C,x}$ is the union of $U_{C,x} = \bigcup_{z \in V((T_C)_x)} P_z$ and all the vertex sets of the connected components C' of $C - S_C$ having a neighbor in $U_{C,x}$.

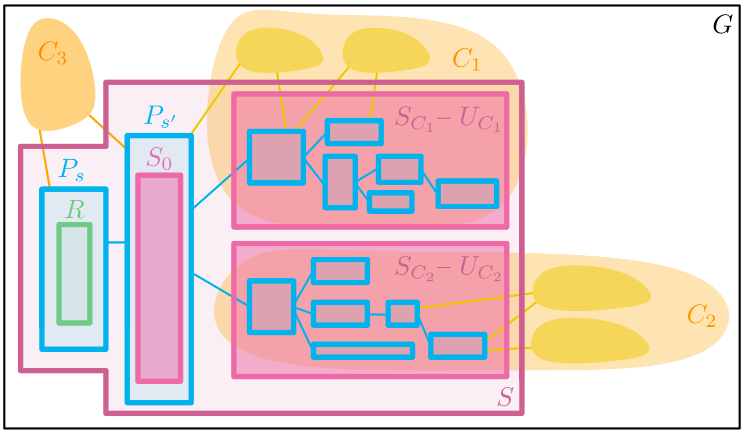


Figure 6.10: An illustration of the proof of Claim 6.3.1. In the sketched case, $C_1, C_2 \in \mathcal{C}_1$ and $C_3 \notin \mathcal{C}_1$.

We set

$$S = R \cup S_0 \cup \bigcup_{C \in \mathcal{C}_1} (S_C \setminus R_C).$$

See Figure 6.10 again. Let T be obtained from the disjoint union of $\{T_C \mid C \in \mathcal{C}_1\}$ by identifying the vertices $\{s_C \mid C \in \mathcal{C}_1\}$ into a new vertex s' and by adding the root s adjacent only to s' in T. Let $P_s = R$, $P_{s'} = S_0$, and for each $C \in \mathcal{C}_1$, $x \in V(T_C \setminus \{s_C\})$, let $P_x = P_{C,x}$.

In order to conclude the proof of the claim, we argue that $(T, (P_x \mid x \in V(T)))$ is a tree partition of (G, S) and that (a)-(c) hold.

Since for every $C \in \mathcal{C}_1$, $R \cap N_G(V(C)) = \emptyset$, every edge in $G[S]$ containing a vertex in R has another endpoint in $R \cup S_0 = P_s \cup P_{s'}$. Consider an edge vw in $G[S]$ such that $v \in S_0$ and $w \in S_C$ for some $C \in \mathcal{C}_1$. Since $(T_C, (P_{C,x} \mid x \in V(T_C)))$ is a tree partition of (G_C, S_C) with $P_{C,s_C} = R_C$ and $S_0 \subseteq V((G-R)-V(C))$, we conclude that $w \in P_x$ for some $x \in V(T_C)$ such that $s'x$ is an edge in T. Finally, for every edge vw of $G[S]$ with $v, w \notin R \cup S_0$, vw is an edge of $G[S_C \setminus R_C]$ for some connected component $C \in \mathcal{C}_1$, and so $v \in P_{C,x}$ and $w \in P_{C,y}$ for adjacent or identical vertices x, y of T_C. Then $v \in P_x$ and $w \in P_y$. For every connected component C' of $G-S$, either $N_G(V(C')) \subseteq U \cup S_0 = P_s \cup P_{s'}$, or $C' \subseteq C$ for some $C \in \mathcal{C}_1$. In the latter case, C' is a connected component of $G_C - S_C$, and $N_G(U) \cap V(C) = \emptyset$. Since $(T_C, (P_{C,x} \mid x \in V(T_C)))$ is a tree partition of (G_C, S_C), there is $x, y \in V(T_C)$ such that $N_{G_C}(V(C')) \subseteq P_{C,x} \cup P_{C,y}$. If $x = s_C$, then $N_G(V(C')) \subseteq P_s \cup P_y$ and $sy \in E(T)$. If $y = s_C$, then $N_G(V(C')) \subseteq P_s \cup P_x$ and $sx \in E(T)$. Finally, if $x, y \neq s_c$, then $N_G(V(C')) \subseteq P_s \cup P_x$ and $xy \in E(T)$. This proves that $(T, (P_x \mid x \in V(T)))$ is a tree partition of (G, S).

Let $F \in \mathcal{F}$. If $V(F) \cap (R \cup S_0) \neq \emptyset$, then $V(F) \cap S \neq \emptyset$. Otherwise, $F \subseteq G - R$ and $V(F) \cap S_0 = \emptyset$, and therefore, by (g1'), we have $F \notin \mathcal{F}'$. In this case, there is a connected component $C \in \mathcal{C}_1$ such that $F \in \mathcal{F}|_C$. By (a'), $V(F) \cap (S_C \setminus R_C) \neq \emptyset$ and so $V(F) \cap S \neq \emptyset$. This proves (a).

Consider a connected component C' of $G - S$. If $C' \subseteq C$ for some $C \in \mathcal{C}_1$, then by (b'), it follows that $N_{G_C}(V(C'))$ intersects at most $2\alpha'$ connected components of $G_C - V(C')$, and so $N_G(V(C'))$ intersects at most $2\alpha'$ connected components of $G - V(C')$. Otherwise, C' is a connected component of $(G - R) - S_0$ such that $N_G(R) \cap V(C') \neq \emptyset$. By (g2'), $N_{G-R}(V(C'))$ intersects at most α' connected components of $(G - R) - V(C')$, and therefore, $N_G(V(C'))$ intersects at most $\alpha' + |R| \leqslant 2\alpha'$ connected components of $G - V(C')$. This proves (b).

Finally, we argue (c). Let $x \in V(T)$. If $x = s$, $P_x = P_s = R$, which has size at most α', and thus the assertion holds by (z2). If $x = s'$, then $G_{s'}$ is a union of connected components of $G - R$, and $P_x = P_{s'} = S_0$, and so, by (z1) and (g3'),

$$\mathrm{par}_q(G_{s'}, S_0) \leqslant \mathrm{par}_q(G - R, S_0) \leqslant \beta'(X'')g(q).$$

If $x \in V(T_C - \{s_C\})$ for some $C \in \mathcal{C}_1$, we have $T_x = (T_C)_x$ and $G_x = G_{C,x}$. Thus, the asserted inequality follows from (c'). This ends the proof of the claim. ♢

Recall that X is a fixed graph in $\mathbf{A}(\mathcal{X})$. Let

$$\alpha = 2\alpha' \quad \text{and} \quad \beta(X) = 2b' \max\{\beta'(X''), \alpha'\}.$$

In order to conclude the proof, we argue that α and $\beta(\cdot)$ witness $q \mapsto q \cdot g(bq)$ being $(\mathrm{par}, \mathcal{X})$-bounding. Let q be a positive integer, let G be a K_k-minor-free graph, and let $\mathcal{F}$ be a family of connected subgraphs of G such that there are no $\mathcal{F}$-rich model of X in G.

First, assume that G is connected. By Claim 6.3.1, applied for R an arbitrary singleton in $V(G)$, there exist $S \subseteq V(G)$, a tree T rooted in $s \in V(T)$, and a tree partition $(T, (P_x \mid x \in V(T)))$ of (G, S) with $P_s = R$ such that (a), (b), and (c) are satisfied. Items

(a) and (b) immediately imply (g1) and (g2), respectively. Additionally, (z4) and (c) imply (g3), since

$$\mathrm{par}_q(G, S) \leqslant b'(q+1) \cdot \max_{x \in V(T)} \mathrm{par}_{bq}(G_x, P_x) \leqslant \beta(X) \cdot qg(q).$$

Finally, assume that G is not connected and let $\mathcal{C}$ be the family of all the connected components of G. Since we have already proved the assertion for connected graphs, for every $C \in \mathcal{C}$, there exists $S_C \subseteq V(C)$ such that

(g1') $V(F) \cap S_C \neq \emptyset$ for every $F \in \mathcal{F}|_C$,

(g2') for every connected component C' of $C - S_C$, $N_C(V(C'))$ intersects at most α connected components of $C - V(C')$, and

(g3') $\mathrm{par}_q(C, S_C) \leqslant \beta(X) \cdot qg(bq)$.

Let $S = \bigcup_{C \in \mathcal{C}} S_C$. To conclude the proof, we show that (g1), (g2), and (g3) hold for G, $\mathcal{F}$, and S. Items (g1) and (g2) follow from (g1') and (g2'), respectively. Additionally, (z1) and (g3') imply (g3), since

$$\mathrm{par}_q(G, S) = \max_{C \in \mathcal{C}} \mathrm{par}_q(C, S_C) \leqslant \beta(X) \cdot qg(bq).$$

Therefore, $q \mapsto qg(bq)$ is $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$-bounding, which concludes the proof. □

We now prove Theorem 6.4.

Proof of Theorem 6.4. Let k be a positive integer. Suppose that g is $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$-bounding. Hence, there exists a positive integer α' such that for every graph $X \in \mathbf{A}(\mathcal{X})$, there is an integer $\beta'(X)$ such that for every positive integer q, for every K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if G has no $\mathcal{F}$-rich model of X, then there exists $S \subseteq V(G)$ such that

(g1') $S \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$;

(g2') for every connected component C of $G - S$, $N_G(V(C))$ intersects at most α' connected components of $G - V(C)$;

(g3') $\mathrm{par}_q(G, S) \leqslant \beta'(X) \cdot g(q)$.

Let α' and $\beta'(\cdot)$ be as above.

Claim 6.3.2. *Let $X \in \mathbf{T}(\mathcal{X})$ connected. There exist positive integers $\alpha_0(X)$ and $\beta_0(X)$ such that for every positive integer q, for every K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if there is no $\mathcal{F}$-rich model of X in G, then there exists $S \subseteq V(G)$ such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,*

(b) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $\alpha_0(X)$ connected components of $G - V(C)$, and*

(c) $\mathrm{par}_q(G, S) \leqslant \beta_0(X) \cdot g(q)$.

Proof of the claim. Let $X \in \mathbf{T}(\mathcal{X})$, and let $\big(T, (W_a \mid a \in V(T))\big)$ be a rooted forest decomposition of X of adhesion at most 1 witnessing the fact that $X \in \mathbf{T}(\mathcal{X})$. Without loss of generality, we assume that $W_a \neq \emptyset$ for every $a \in V(T)$. Since X is connected, this implies that T is a tree. We proceed by induction on the vertex-height of T. If $T = K_1$, then the result is clear. Now suppose $T \neq K_1$.

We denote by r the root of T, and let $z \in V(X)$ such that $W_r = \{z\}$. Let T_0 be the subtree of T induced by $\{a \in V(T) \mid z \in W_a\}$. Observe that for every $a \in V(T_0)$, $X[W_a \setminus \{z\}] \in \mathcal{X}$. Since $\mathcal{X}$ is closed under disjoint union, this implies that $X[\bigcup_{a \in V(T_0)} W_a \setminus \{z\}] \in \mathcal{X}$. Let

$$X_0 = X\left[\textstyle\bigcup_{a \in V(T_0)} W_a\right],$$

and let X_1 be obtained from X_0 by adding for every $x \in V(X_0)$ a leaf ℓ_x with neighbor x. Then, since $\mathcal{X}$ is stable under leaf addition, $X_1 \setminus \{z\} \in \mathcal{X}$, and so $X_1 \in \mathbf{A}(\mathcal{X})$.

For every $x \in V(X_0)$, let $\mathcal{R}_x$ be family of all the connected components R of $T - V(T_0)$ such that $x \in \bigcup_{a \in V(R)} W_a$, and let

$$X_x = \begin{cases} X\left[\bigcup_{R \in \mathcal{R}_x, a \in V(R)} W_a\right] & \text{if } \mathcal{R}_x \neq \emptyset, \\ X[\{x\}] & \text{if } \mathcal{R}_x = \emptyset. \end{cases}$$

Now, let X_2 be obtained from X by identifying all the vertices in $V(X_0)$ into a single vertex z_2. Note that for every $x \in V(X_0)$, X_2 contains a copy of X_x with x relabeled into z_2. Then, let X_3 be obtained from the union of two copies X_2^1 and X_2^2 of X_2 that intersect only in z_2.

We now build a rooted tree decomposition $\big(T', (W'_a \mid a \in V(T'))\big)$ of X_3 witnessing the fact that $X_3 \in \mathbf{T}(X)$, and with T' of smaller vertex-height than T. Let T'_0 be obtained from T by contracting T_0 into a single vertex r', and then let T' be the union of two copies $T_0'^1$ and $T_0'^2$ of T'_0 intersecting only in r'. Then, for every $a \in V(T_0'^i)$ for $i \in \{1,2\}$, let W'_a be the union of $\{z_2\}$ with the image of $W_a \setminus V(X_0)$ in the copy X_2^i of X_2. Then $\big(T', (W_a \mid a \in V(T'))\big)$ is a rooted tree decomposition of X_3 witnessing the fact that $X_3 \in \mathbf{T}(\mathcal{X})$. We claim that the vertex-height of T' is smaller than the vertex-height of T. Indeed, for every neighbor s of r in T, since X is connected and $W_s \neq \emptyset$, we have $z \in W_s$, and so $s \in V(T_0)$. This proves that $N_T(r) \subseteq V(T_0)$, and so T' has smaller vertex-height than T. Therefore, we can call the induction hypothesis on X_3. Let

$$\alpha_0(X) = \alpha' + \alpha_0(X_3) \quad \text{and} \quad \beta_0(X) = \beta'(X_1) + \beta_0(X_3).$$

Let q be a positive integer. Let G be a K_k-minor-free graph, and let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of X in G. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of G such that there is an $\mathcal{F}|_H$-rich model of X_3 in H.

We claim that there is no $\mathcal{F}'$-rich model of X_1 in G. Suppose for contradiction that there is an $\mathcal{F}'$-rich model $(A_x \mid x \in V(X_0))$ of X_1 in G. Then, for every $x \in V(X_0)$, the graph $G[A_{\ell_x}]$ contains a member of $\mathcal{F}'$ and so an $\mathcal{F}$-rich model $(B_{x,y} \mid y \in V(X_3))$ of X_3. Since $G[A_{\ell_x}]$ is connected, there is an $(N(A_x), B_{x,y})$-path P in $G[A_{\ell_x}]$ for some $y \in V(X_3)$. Recall that X_3 consists of two copies X_2^1 and X_2^2 of X_2 that intersect only in z_2, and X_2 is connected. Therefore, without loss of generality, say that $y \in V(X_2^2)$. By replacing B_{x,z_2} by the union $V(P) \cup \bigcup_{y \in V(X_2^2)} B_{x,y}$, we obtain a model $(C_{x,y} \mid y \in V(X_2))$ of X_2 in $G[A_{\ell_x}]$ such that C_{x,z_2} contains a neighbor of a vertex in A_x. By restricting this model to $(V(X_x) \setminus \{x\}) \cup \{z_2\}$, and relabeling z_2 by x, we obtain a model $(D_{x,y} \mid y \in V(X_x))$ of X_x in $G[A_{\ell_x}]$ such that $D_{x,x}$ contains a neighbor of A_x. Finally, for every $y \in V(X)$, let

$$E_y = \begin{cases} A_y \cup D_{y,y} & \text{if } y \in V(X_0), \\ D_{x,y} & \text{if } y \notin V(X_0), \text{ for } x \in V(X_0) \text{ such that } y \in V(X_x). \end{cases}$$

Then, $(E_y \mid y \in V(X))$ is an $\mathcal{F}$-rich model of X in G, a contradiction. This proves that G has no $\mathcal{F}'$-rich model of X_1 in G.

By the definition of β' applied to X_1, G, $\mathcal{F}'$, there exists $S_0 \subseteq V(G)$ such that

(g1") $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}'$;

(g2") for every connected component C of $G - S_0$, $N_G(V(C))$ intersects at most α' connected components of $G - V(C)$;

(g3") $\mathrm{par}_q(G, S_0) \leqslant \beta'(X_1) \cdot g(q)$

Since $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}'$, $G - S_0$ has no $\mathcal{F}|_{G-S_0}$-rich model of X_3. Therefore, by the induction hypothesis, there exists $S_1 \subseteq V(G - S_0)$ such that

(a') $V(F) \cap S_A \neq \emptyset$ for every $F \in \mathcal{F}|_{G-S_0}$;

(b') for every connected component C of $G - S_0 - S_1$, $N_{G-S_0}(V(C))$ intersects at most $\alpha_0(X_3)$ connected components of $G - S_0 - V(C)$;

(c') $\mathrm{par}_q(G - S_0, S_1) \leqslant \beta_0(X_3) \cdot g(q)$.

Finally, let

$$S = S_0 \cup S_1.$$

We now argue (a), (b), and (c). First, observe that for every $F \in \mathcal{F}$, either $V(F) \cap S_0 \neq \emptyset$, or $F \in \mathcal{F}|_{G-S_0}$ and so $V(F) \cap S_1 \neq \emptyset$ by (a'). This proves (a).

Then, for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $\alpha_0(X_3)$ connected components of $G - S_0 - V(C)$ by (b'), and at most α' connected components of $G - S_0$. Therefore, $N_G(V(C))$ intersects at most $\alpha' + \alpha_0(X_3) = \alpha_0(X)$ connected components of $G - V(C)$, which proves (b).

Finally,

$$\begin{aligned} \mathrm{par}_q(G, S) &\leqslant \mathrm{par}_q(G, S_0) + \mathrm{par}_q(G - S_0, S_1) && \text{by (z3)} \\ &\leqslant \beta'(X_1) \cdot g(q) + \beta_0(X_3) \cdot g(q) && \text{by (g3") and (c')} \\ &\leqslant \beta_0(X) \cdot g(q). \end{aligned}$$

This proves (c). ◊

Claim 6.3.2 implies that g is weakly $(\mathrm{par}, \mathbf{T}(\mathcal{X}))$-bounding. Since $\mathcal{X}$ is nonempty and closed under leaf addition, every edgeless graph is a subgraph of a graph in $\mathcal{X} \subseteq \mathbf{A}(\mathcal{X})$. Therefore, since g is $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$-bounding, we deduce that g is $(\mathrm{par}, \mathcal{E})$-bounding. It follows by Lemma 6.22 that g is $(\mathrm{par}, \mathbf{T}(\mathcal{X}))$-bounding. This proves the theorem. □

6.6 Common arguments for the base cases

In this section, we discuss and prove some statements that are common for some of the final proofs. In Section 6.6.1, we prove a very general tool to prove that a function is $(\mathrm{par}, \mathcal{S}_2)$-bounding for any nice family of focused parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$. In Section 6.6.2, we prove structural properties that will be used in Sections 6.7 to 6.9 to find $(\mathrm{par}, \mathcal{R}_2)$-bounding functions, for some specific families of focused parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$.

6.6.1 Excluding a linear forest

The following theorem will be one of the base cases in the proofs of Theorems 1.34 to 1.36. Recall that $\mathcal{E}$ is the class of all edgeless graphs, and $\mathcal{S}_2$ is the class of all linear forests.

Theorem 6.23. *Let* par *be a nice family of focused parameters, and let* $g\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$. *If* g *is* $(\mathrm{par}, \mathcal{E})$*-bounding, then* g *is* $(\mathrm{par}, \mathcal{S}_2)$*-bounding.*

This proof is a modification of an argument present in [PPT23b, Theorem 3.22].

Proof of Theorem 6.23. Let k be a positive integer. For every positive integer d, applying the hypothesis that g is $(\mathrm{par}, \mathcal{E})$-bounding for $X = \overline{K_d}$, we obtain that there exists α' and $\beta'(d)$ such that the following holds.

For every positive integer q, for every K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if there are no d pairwise disjoint members of $\mathcal{F}$, then there exists $S \subseteq V(G)$ such that

(g1') $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,

(g2') for every connected component C of $G - S$, $N_G(V(C))$ intersects at most α' connected components of $G - V(C)$, and

(g3') $\mathrm{par}_q(G, S) \leqslant \beta'(d) \cdot g(q)$.

Let $X \in \mathcal{S}_2$. Note that X is a subgraph of $P_{|V(X)|} \in \mathcal{S}_2$, and so, without loss of generality, $X = P_\ell$ for some positive integer ℓ. We suppose that the vertices of P_ℓ are $1, \dots, \ell$, in this order along P_ℓ.

We now show by induction on ℓ that for every positive integer q, for every K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if there is no $\mathcal{F}$-rich model of P_ℓ in G, then there exists $S \subseteq V(G)$ such that

(a) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,

(b) for every connected components C of $G - S$, $N_G(V(C))$ intersects at most $(\ell - 1)\alpha'$ connected components of $G - V(C)$, and

(c) $\mathrm{par}_q(G, S) \leqslant (\ell - 1) \cdot g(q)$.

Let G be a K_k-minor-free graph and let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of P_ℓ in G. If $\ell = 1$, then $\mathcal{F}$ is empty and this clearly holds for $S = \emptyset$ by (z2). Now suppose $\ell > 1$ and that the result holds for $\ell - 1$.

Let $\mathcal{C}$ be the family of all the connected components of G, and let $C \in \mathcal{C}$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of C such that H contains an $\mathcal{F}$-rich model of $P_{\ell-1}$. We claim that there are no two disjoint members of $\mathcal{F}'$ in C. Indeed, if $(A_1, \ldots, A_{\ell-1})$ and $(B_1, \ldots, B_{\ell-1})$ are two disjoint $\mathcal{F}$-rich models of $P_{\ell-1}$ in C, then, because C is connected, there is a path Q between A_i and B_j for some $i, j \in [\ell-1]$ which is internally disjoint from $\bigcup_{k\in[\ell-1]} A_k \cup \bigcup_{k\in[\ell-1]} B_k$. By possibly reversing the orderings $(A_1, \ldots, A_{\ell-1})$ and $(B_1, \ldots, B_{\ell-1})$, we assume that $i \geqslant \frac{\ell-1}{2}$ and $j \leqslant \frac{\ell-1}{2}$. Then, $(A_1, \ldots, A_i, (V(Q) \setminus A_i) \cup B_j, \ldots, B_{\ell-1})$ is an $\mathcal{F}$-rich model of $P_{\ell'}$ in C, where $\ell' = i + (\ell - j) \geqslant \frac{\ell-1}{2} + \ell - \frac{\ell-1}{2} = \ell$. This contradicts the fact that G has no $\mathcal{F}$-rich model of P_ℓ. Therefore, there are no two disjoint members of $\mathcal{F}'$.

Hence, there exists $S_{0,C} \subseteq V(C)$ such that

(g1”) $V(F) \cap S_{0,C} \neq \emptyset$ for every $F \in \mathcal{F}'$,

(g2”) for every connected component C' of $C - S_{0,C}$, $N_C(V(C'))$ intersects at most α' connected components of $G - V(C)$, and

(g3”) $\mathrm{par}_q(C, S_{0,C}) \leqslant \beta'(2) \cdot g(q)$.

Now, by (g1”), there is no $\mathcal{F}$-rich model of $P_{\ell-1}$ in $C - S_{0,C}$. Therefore, by the induction hypothesis applied to $C - S_{0,C}$ and $\mathcal{F}|_{C-S_{0,C}}$, there exists $S_{1,C} \subseteq V(C)$ such that

(a’) $V(F) \cap S_{1,C} \neq \emptyset$ for every $F \in \mathcal{F}|_{C-S_{0,C}}$,

(b’) for every connected component C' of $C - (S_{0,C} \cup S_{1,C})$, $N_{C-S_{0,C}}(V(C'))$ intersects at most $(\ell-2)\alpha'$ connected components of $C - S_{0,C}$.

(c’) $\mathrm{par}_q(C - S_{0,C}, S_{1,C}) \leqslant (\ell-2)\beta'(2) \cdot g(q)$.

Now, let

$$S = \bigcup_{C\in\mathcal{C}} (S_{0,C} \cup S_{1,C}).$$

We now argue (a), (b), and (c). First, for every $F \in \mathcal{F}$, since F is connected, there exists $C \in \mathcal{C}$ such that $F \subseteq V(C)$. Therefore, either $V(F) \cap S_{0,C} \neq \emptyset$ and so $V(F) \cap S \neq \emptyset$, or $C \in \mathcal{F}|_{C-S_{0,C}}$, but then $V(F) \cap S_{1,C} \neq \emptyset$ by (a’), and so $V(F) \cap S \neq \emptyset$. This proves (a).

Second, for every connected component C' of $G - S$, there exists $C \in \mathcal{C}$ such that C is a connected component of $C - S_{0,C} - S_{1,C}$. Then, by (g2”) and (b’), $N_G(V(C))$ intersects at most $\alpha' + (\ell-2)\alpha' = (\ell-1)\alpha'$ connected components of $G - V(C')$. This proves (b).

Finally,

$$\begin{aligned}
\mathrm{par}_q(G, S) &\leqslant \max_{C\in\mathcal{C}} \mathrm{par}_q(C, S_{0,C} \cup S_{1,C}) && \text{by (z1)}\\
&\leqslant \max_{C\in\mathcal{C}} \Big(\mathrm{par}_q(C, S_{0,C}) + \mathrm{par}_q(C - S_{0,C}, S_{1,C})\Big) && \text{by (z3)}\\
&\leqslant (\ell-1)\beta(2) \cdot g(q) && \text{by (g3”) and (c’).}
\end{aligned}$$

This proves (c).

Altogether, this proves that g is weakly $(\mathrm{par}, \mathcal{S}_2)$-bounding. Finally, since g is $(\mathrm{par}, \mathcal{E})$-bounding, we conclude by Lemma 6.22 that g is $(\mathrm{par}, \mathcal{S}_2)$-bounding. □

Theorem 6.24. *Let* par *be a nice family of focused parameters, and let* $g\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$*. If* g *is* $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$*-bounding, then there exists a positive integer* b *such that* $q \mapsto qg(bq)$ *is* $(\mathrm{par}, \mathcal{S}_3)$*-bounding.*

Proof. Let k be a positive integer. Since g is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-bounding, there exists α' and $\beta'(\cdot)$ such that the following holds. For every $X \in \mathbf{A}(\mathcal{S}_2)$, for every K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, for every positive integer q, if there is no $\mathcal{F}$-rich model of X in G, then there exists $S \subseteq V(G)$ such that

(g1') $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,

(g2') for every connected component C of $G - S$, $N_G(V(C))$ intersects at most α' connected components of $G - V(C)$, and

(g3') $\mathrm{par}_q(G, S) \leqslant \beta'(X) \cdot g(q)$.

Claim 6.24.1. *Let* X *be a connected graph in* $\mathcal{S}_3$*, there exists positive integers* $\alpha_0(X), \beta_0(X)$ *such that the following holds. Let* q *be a positive integer, let* G *be a* K_k*-minor-free graph, let* $\mathcal{F}$ *be a family of connected subgraphs of* G *such that there is no* $\mathcal{F}$*-rich model of* X *in* G*. There exists* $S \subseteq V(G)$ *such that*

(a) $V(F) \cap S \neq \emptyset$ *for every* $F \in \mathcal{F}$*,*

(b) *for every connected component* C *of* $G - S$*,* $N_G(V(C))$ *intersects at most* $\alpha_0(X)$ *connected components of* $G - V(C)$*,*

(c) $\mathrm{par}_q(G, S) \leqslant \beta_0(X) \cdot g(q)$.

Proof of the claim. Let $(T, (W_a \mid a \in V(T)))$ be a rooted forest decomposition of X witnessing the fact that $X \in \mathcal{S}_3$. By taking such a rooted forest decomposition with $|V(T)|$ minimum, and because X is connected, T is a rooted tree. We will proceed by induction on the vertex-height of T. If $T = K_1$, then $X \in \mathbf{A}(\mathcal{S}_2)$ and so the result holds for $\alpha_0(X) = \alpha'$ and $\beta_0(X) = \beta'(X)$. Now suppose $T \neq K_1$.

Let T_0 be the subtree of T induced by the internal vertices of T, that is the vertices of T which are either the root of not leaves, and let $X_0 = X[\bigcup_{a \in V(T_0)} W_a]$. For every $x \in V(X_0)$, let

$$X_x = X\left[\bigcup_{a \in V(T) \setminus V(T_0), x \in W_a} W_a\right].$$

Note that $X_x \in \mathbf{A}(\mathcal{S}_2)$ for every $x \in V(X_0)$. Let ℓ be the maximum of $|V(X_x)| - 1$ over all $x \in V(T_0)$.

Let

$$\alpha_0(X) = \alpha_0(X_0) + \alpha' \quad \text{and} \quad \beta_0(X) = \beta_0(X_0) + \beta'(K_1 \oplus P_{4\ell}).$$

Let q be a positive integer, let G be a K_k-minor-free graph, let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of X.

Let $\mathcal{F}'$ be the family of all the connected subgraphs H of G such that H contains an $\mathcal{F}$-rich model of $K_1 \oplus P_{4\ell}$. We claim that there is no $\mathcal{F}'$-rich model of X_0 in G. To see that, suppose for sake of a contradiction that $(B_x \mid x \in V(X_0))$ is an $\mathcal{F}'$-rich model of X_0 in G. Let $x \in V(X_0)$. The subgraph $G[B_x]$ contains an $\mathcal{F}$-rich model $(C_{x,y} \mid y \in V(K_1 \oplus P_{4\ell}))$ of $K_1 \oplus P_{4\ell}$. We label the vertices of $P_{4\ell}$ by $1, \dots, 4\ell$, in this order. Observe that x has degree 3 in X_0. Let $x_1, \dots, x_d$

be the neighbors of x in X_0. For every $i \in [d]$, B_x contains a neighbor of B_{x_i}, and since $G[B_x]$ is connected, there is a path Q_i from a neighbor of B_{x_i} to C_{x,y_i} in $G[B_x]$ for some $y_i \in V(K_1 \oplus P_{4\ell})$, with Q_i internally disjoint from $\bigcup_{y \in V(K_1 \oplus P_{4\ell})} C_{x,y}$. Since $d \leqslant 3$, by the pigeonhole principle, there is an interval $I \subseteq [4\ell]$ of size ℓ such that $I \cap \{y_1, \dots, y_d\} = \emptyset$. Now, relabeling the branch sets $(C_{x,y} \mid y \in I), \bigcup_{i \in [d]} V(Q_i) \cup \bigcup_{y \in V(K_1 \oplus P_{4\ell}) \setminus I} C_{x,y}$, we obtain an $\mathcal{F}$-rich model $(D_{x,y} \mid y \in V(K_1 \oplus P_\ell))$ of $K_1 \oplus P_\ell$ in $G[B_x]$, such that $D_{x,z}$ contains a neighbor of B_{x_i} for each $i \in [d]$, where z is the vertex of K_1 in $K_1 \oplus P_\ell$. Now, since $\ell \geqslant |V(X_x)| - 1$, the graph $K_1 \oplus P_\ell$ contains X_x as a subgraph. Therefore, we obtain an $\mathcal{F}$-rich model $(E_{x,y} \mid y \in V(X_x))$ of X_x in $G[B_x]$ such that $E_{x,x}$ contains a neighbor of B_y for every $y \in N_X(x)$.

Now, for every $x \in V(X)$, let

$$F_x = \begin{cases} E_{x',x} & \text{if } x \notin V(X_0) \text{ and } x' \in V(X_0) \text{ is such that } x \in V(X_{x'}), \\ B_x \cup E_{x,x} & \text{if } x \in V(X_0). \end{cases}$$

Then, $(F_x \mid x \in V(X))$ is an $\mathcal{F}$-rich model of X in G, a contradiction. This proves that there is no $\mathcal{F}'$-rich model of X_0 in G.

Since $(T_0, (W_a \mid a \in V(T_0)))$ is a tree decomposition of X_0 witnessing the fact that $X_0 \in \mathbf{T}(\mathcal{S}_2)$, and because T_0 has smaller vertex-height than T, we can apply induction. Therefore, there exists $S_0 \subseteq V(G)$ such that

(a') $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}'$,

(b') for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $\alpha_0(X_0)$ connected components of $G - V(C)$, and

(c') $\mathrm{par}_q(G, S) \leqslant \beta_0(X_0) \cdot g(q)$.

Now, by (a'), there is no $\mathcal{F}$-rich model of $K_1 \oplus P_{4\ell}$ in $G - S_0$. Therefore, there exists $S_1 \subseteq V(G - S_0)$ such that

(g1") $V(F) \cap S_1 \neq \emptyset$ for every $F \in \mathcal{F}|_{G - S_0}$,

(g2") for every connected component C of $G - S_0 - S_1$, $N_{G-S_0}(V(C))$ intersects at most α' connected components of $G - S_0 - V(C)$, and

(g3") $\mathrm{par}_q(G - S_0, S_1) \leqslant \beta'(K_1 \oplus P_{4\ell}) \cdot g(q)$.

Now, let

$$S = S_0 \cup S_1.$$

First, for every $F \in \mathcal{F}$, either $V(F) \cap S_0 \neq \emptyset$, or $F \in \mathcal{F}|_{G-S_0}$ and so $V(F) \cap S_1 \neq \emptyset$ by (g1"). In both cases, $V(F) \cap S \neq \emptyset$. This proves (a).

Second, for every connected component G of $G - S$, there is a connected component C' of $G - S_0$ containing C. Since $N_G(V(C))$ intersects at most $\alpha_0(X_0)$ connected components of $G - V(C')$ by (b'), and $N_{G-S_0}(V(C))$ intersects at most α' connected components of $G - S_0 - V(C)$ by (g2"), we conclude that $N_G(V(C))$ intersects at most $\alpha_0(X_0) + \alpha' = \alpha_0(X)$ connected components of $G - V(C)$. This proves (b).

Finally,

$$\begin{aligned}\mathrm{par}_q(G,S) &\leqslant \mathrm{par}_q(G,S_0)+\mathrm{par}_q(G,S_1) && \text{by (z3)},\\ &\leqslant (\beta_0(X_0)+\beta'(X_1\oplus P_{4\ell}))\cdot g(q) && \text{by (c') and (g3")},\\ &= \beta_0(X)\cdot g(q).\end{aligned}$$

This shows (c), and concludes the proof of the claim. ◊

Claim 6.24.1 implies that g is weakly $(\mathrm{par},\mathcal{S}_3)$-bounding par. Since $\mathcal{E}\subseteq\mathbf{A}(\mathcal{S}_2)$, g is $(\mathrm{par},\mathcal{E})$-bounding. By Lemma 6.22, this implies that g is $(\mathrm{par},\mathcal{S}_3)$-bounding, which concludes the proof of the theorem. □

6.6.2 Excluding a forest

In this section, we show structural properties for graphs with no $\mathcal{F}$-rich model of a given forest. This will be typically used in Sections 6.7 and 6.8 to show that $q\mapsto\log q$ is $(\mathrm{par},\mathcal{R}_2)$-bounding, and in Section 6.9 to show that $q\mapsto q\log q$ is $(\mathrm{par},\mathcal{R}_3)$-bounding, for some families of focused parameters $(\mathrm{par}_q\mid q\in\mathbb{N}_{>0})$. The main inspiration for this section is the paper [DHJ+23] by Dujmović, Hickingbotham, Joret, Micek, Morin, and Wood.

Let h,d be positive integers. We denote by $F_{h,d}$ the (rooted) complete d-ary tree of vertex-height h. In particular, $F_{2,d}$ is the star with d leaves. Note that for every forest X, there exist positive integers h,d such that X is a subgraph of $F_{h,d}$.

Lemma 6.25. *Let h,d be positive integers. Let G be a graph, let $\mathcal{D}=\big(T,(W_x\mid x\in V(T))\big)$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{h+1,d}$. There exist pairwise disjoint $S_1,\dots,S_h\subseteq V(G)$, and for every $a\in[h]$ a path partition $(P_{a,0},\dots,P_{a,\ell_a})$ of $(G-S_1-\dots-S_{a-1},S_a)$ such that for $S=\bigcup_{a\in[h]}S_a$,*

(a) *$V(F)\cap S\neq\emptyset$ for every $F\in\mathcal{F}$;*

(b) *for every $a\in[h]$, for every $i\in[\ell_a]$, $P_{a,i}$ is contained in a union of at most $d+h-1$ bags of $\mathcal{D}$.*

Proof. We proceed by induction on h. For $h=1$, the result follows from Lemma 3.14. Now suppose $h>1$ and that the result holds for $h-1$.

Let $\mathcal{F}'$ be the family of all the connected subgraphs H of G containing an $\mathcal{F}$-rich model of $F_{h-1,d+1}$. We claim that there is no $\mathcal{F}'$-rich model of $F_{2,d}$ in G.

Suppose for a contradiction that there is an $\mathcal{F}'$-rich model $\big(B_x\mid x\in V(F_{2,d})\big)$ of $F_{2,d}$ in G. We will deduce an $\mathcal{F}$-rich model of $F_{h,d}$ in G. We denote by c the center of the star $F_{2,d}$ and by $x_1,\dots,x_d$ the leaves of $F_{2,d}$. We denote by r the root of $F_{h-1,d+1}$.

Let $i\in[d]$. Since $(B_x\mid x\in V(F_{2,d}))$ is $\mathcal{F}'$-rich, B_{x_i} contains a neighbor u_i of B_c, and $G[B_{x_i}]$ has an $\mathcal{F}|_H$-rich model $(B^i_x\mid x\in V(F_{h-1,d+1}))$ of $F_{h-1,d+1}$. By Lemma 3.16 there is an $\mathcal{F}$-rich model $\mathcal{M}_i$ of $F_{h-1,d}$ in $G[B_{x_i}]$ whose branch set of the root contains a neighbor of B_c.

It follows that the union of the models $\mathcal{M}_1,\dots,\mathcal{M}_d$, together with B_c for the branch set of the root, yields an $\mathcal{F}$-rich model of $F_{h,d}$ in G, a contradiction. This proves that there is no $\mathcal{F}'$-rich model of $F_{2,d}$ in G.

Therefore, by Lemma 3.14 applied for an arbitrary u, there is a set $S_1\subseteq V(G)$ and a path partition $(P_0,\dots,P_\ell)$ of $G[S_1]$ such that

3.14.(a) $V(F) \cap S_1 \neq \emptyset$ for every $F \in \mathcal{F}'$;

3.14.(b) for every connected component C of $G - S_1$, $N_G(V(C)) \subseteq P_i \cup P_j$ for some $i, j \in \{0, \dots, \ell\}$ with either $i = j$ or $j = i + 1$;

3.14.(c) for every $i \in [\ell]$, P_i is contained in the union of at most d bags of $\mathcal{D}$.

By 3.14.(a), $G - S_1$ has no $\mathcal{F}$-rich model of $F_{h-1,d+1}$. Therefore, by the induction hypothesis, there exist pairwise disjoint $S'_1, \dots, S'_h \subseteq V(G)$, and for every $a \in [h]$ there is a path partition $(P'_{a,0}, \dots, P'_{a,\ell_a})$ of $(G - S_1 - S'_1 - \dots - S'_{a-1}, S'_a)$ such that for $S' = \bigcup_{a \in [h]} S'_a$,

(a') $V(F) \cap S' \neq \emptyset$ for every $F \in \mathcal{F}|_{G-S_1}$;

(b') for every $a \in [h]$, for every $j \in [\ell_a]$, $P_{a,j}$ is contained in the union of at most $d + h - 1$ bags of $\mathcal{D}$.

Now, for every $a \in [h]$, let $S_{a+1} = S'_a$, and let $(P_{a+1,0}, \dots, P_{a+1,\ell_{a+1}}) = (P'_{a,0}, \dots, P'_{a,\ell_a})$. Finally, let $(P_{1,0}, \dots, P_{1,\ell_1}) = (P_0, \dots, P_\ell)$. Then, $S_1, \dots, S_{h+1}$ is as claimed, which concludes the proof of the lemma. □

Lemma 6.26. *Let h, d be positive integers. Let G be a graph, let $\mathcal{D} = (T, (W_x \mid x \in V(T)))$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{h+1,d}$. There exist pairwise disjoint $S_1, \dots, S_{h+1} \subseteq V(G)$, and for every $a \in [h+1]$ a path partition $(P_{a,0}, \dots, P_{a,\ell_a})$ of $(G - S_1 - \dots - S_{a-1}, S_a)$ such that for $S = \bigcup_{a \in [h+1]} S_a$,*

(a) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;

(b) for every connected component C of $G - S$, $N_G(V(C))$ intersects at most two connected components of $G - S$;

(c) for every $a \in [h+1]$, for every $i \in [\ell_a]$, $P_{a,i}$ is contained in the union of at most $2h(d+h-1)$ bags of $\mathcal{D}$.

Proof. We proceed by induction on $|V(G)|$.

First suppose that G is not connected. Let $\mathcal{C}$ be the family of all the connected components of G, and let $C \in \mathcal{C}$. By the induction hypothesis, there exist pairwise disjoint $S_{C,1}, \dots, S_{C,h+1} \subseteq V(C)$, and for every $a \in [h+1]$ a path partition $(P_{C,a,0}, \dots, P_{C,a,\ell_{C,a}})$ of $(C - S_{C,1} - \dots - S_{C,a-1}, S_{C,a})$ such that for $S_C = \bigcup_{a \in [h+1]} S_{C,a}$,

(a') $V(F) \cap S_C \neq \emptyset$ for every $F \in \mathcal{F}|_C$;

(b') for every connected component C' of $C - S_C$, $N_C(V(C'))$ intersects at most two connected components of $C - S_C$;

(c') for every $a \in [h+1]$, for every $i \in [\ell_{C,a}]$, $P_{C,a,i}$ is contained in the union of at most $2h(d+h-1)$ bags of $\mathcal{D}|_C$.

Then, for every $a \in [h]$, let

$$S_a = \bigcup_{C \in \mathcal{C}} S_{C,a},$$

and let $(P_{a,0}, \dots, P_{a,\ell_a})$ be a concatenation of $(P_{C,a,0}, \dots, P_{C,a,\ell_{C,a}})$ over all $C \in \mathcal{C}$. These objects satisfy the outcome of the lemma.

Now suppose G connected. By Lemma 5.3, we can assume that $\mathcal{D}$ is a natural tree decomposition. By Lemma 6.25, there exist pairwise disjoint $S_1, \dots, S_h \subseteq V(G)$, and for every $a \in [h-1]$ a path partition $(P_{a,0}, \dots, P_{a,\ell_a})$ of $(G - S_1 - \dots - S_{a-1}, S_a)$ such that for $S' = \bigcup_{a \in [h-1]} S_a$,

6.25.(a) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;

6.25.(b) for every $a \in [h]$, for every $i \in [\ell_a]$, $P_{a,i}$ is contained in the union of at most $d+h-1$ bags of $\mathcal{D}$.

Let C' be a connected component of $G - S'$. By 6.25.(b), $N_G(V(C'))$ is included in the union of at most $h(d+h-1)$ bags of $\mathcal{D}$. By Lemma 5.4, there is a set $Z_{C'} \subseteq V(G)$ containing $N_G(V(C'))$ which is included in the union of at most $2h(d+h-1)$ bags of $\mathcal{D}$ such that for every connected component C of $G - Z_{C'}$, $N_G(V(C))$ intersects at most two connected components of $G - V(C)$. Let $S_{C',h+1} = Z_{C'} \cap V(C')$.

Then let

$$S_{h+1} = \bigcup_{C' \in \mathcal{C}} S_{C',h+1}$$

where $\mathcal{C}$ is the family of all the connected component of $G - S'$, and let $S = \bigcup_{a \in [h+1]} S_a$.

First, since $S' \subseteq S$, $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, which proves (a). Second, for every connected component C of $G - S$, there exists $C' \in \mathcal{C}$ such that $C \subseteq C'$, and so C is a connected component of $G - Z_{C'}$, which implies that $N_G(V(C))$ intersects at most two connected components of $G - V(C)$, and so (b) holds. Finally, for an arbitrary ordering $C_1, \dots, C_{\ell_{h+1}}$ of the members of $\mathcal{C}$, $(S_{C_1,h+1}, \dots, S_{C_{\ell_{h+1}},h+1})$ is a path partition of $(G - S', S_{h+1})$, and for every $i \in [\ell_{h+1}]$, $S_{C_i,h+1}$ is included in the union of at most $2h(d+h-1)$ bags of $\mathcal{D}$. This proves (c) and concludes the proof of the lemma. □

6.7 Fractional treedepth-fragility rates

In this section, we prove Theorem 1.36. We start with Theorem 1.33, which is the special corresponding to excluding K_t as a minor. Its proof is an instructive example.

6.7.1 K_t-minor-free graphs

The first ingredient to prove Theorem 1.33 is a bound in the bounded treewidth case due to Dvořák and Sereni.

Theorem 6.27 ([DS20]). *For every positive integer w, there exists a constant $c_{6.27}(w)$ such that for every graph G with $\mathrm{tw}(G) \leqslant w$ and for every positive integer q*

$$\mathrm{ftdfr}_q(G) \leqslant c_{6.27}(w) \cdot q^w.$$

Theorem 6.27 can be used to obtain a good upper-bound on the q-th fractional treedepth-fragility rate of K_t-minor-free graphs of bounded treewidth. To this end, we need the following result of Illingworth, Scott, and Wood.

Theorem 6.28 ([ISW24]). *For every integer t with $t \geqslant 2$ and for every K_t-minor-free graph G with $\operatorname{tw}(G) \leqslant w$ admits a vertex-partition $\mathcal{P}$ such that $G/\mathcal{P}$ has treewidth at most $t-2$ and each part of $\mathcal{P}$ has at most $w+1$ elements*[5].

Theorems 6.27 and 6.28 imply that for every K_t-minor-free graph G of bounded treewidth, we have $\operatorname{ftdfr}_q(G) = \mathcal{O}(q^{t-2})$ as we show below. Let t and w be positive integers, let G be a K_t-minor-free with $\operatorname{tw}(G) \leqslant w$ and let $\mathcal{P}$ be a partition of $V(G)$ as in Theorem 6.28. Since $\operatorname{tw}(G/\mathcal{P}) \leqslant t-2$, we have $\operatorname{ftdfr}_q(G) = \mathcal{O}(q^{t-2})$ by Theorem 6.27. In particular, there exists a random variable Z' over subsets of $\mathcal{P}$ such that $\operatorname{td}(G/\mathcal{P} - \mathsf{Z}') = \mathcal{O}(q^{t-2})$, and Z' is equipped with a q-thin probability distribution. Let $\mathsf{Z} = \bigcup \mathsf{Z}'$. Since every element of $\mathcal{P}$ has at most $w+1$ vertices of G, $\operatorname{td}(G - \mathsf{Z}) \leqslant (w+1) \cdot \operatorname{td}(G/\mathcal{P} - \mathsf{Z}')$. It follows that Z witnesses that $\operatorname{ftdfr}_q(G) = \mathcal{O}((w+1) \cdot q^{t-2})$.

Lifting this reasoning to the general K_t-minor-free case requires the following technical lemma, which we prove at the end of this section.

Lemma 6.29. *For all positive integers t, a, there exists a positive integer $c_{6.29}(t)$ such that for every K_t-minor-free graph G and every positive integer q, there exists a random variable Y over subsets of $V(G)$ such that $\operatorname{tw}(G - \mathsf{Y}) \leqslant c_{6.29}(t) \cdot q$, and Y is equipped with a q-thin probability distribution.*

Proof of Theorem 1.33. Let t be an integer with $t \geqslant 2$, let $c_{6.29}(t)$ be the constant as in Lemma 6.29, let $c_{6.27}(t-2)$ be the constant as in Theorem 6.27. We set $c = 2^t \cdot c_{6.29}(t) \cdot c_{6.27}(t-2)$. Let G be a K_t-minor-free graph and let q be a positive integer. By Lemma 6.29, there exists a random variable Y over subsets Y of $V(G)$ such that $\operatorname{tw}(G - Y) \leqslant c_{6.29}(t) \cdot 2q$, and Y is equipped with a $2q$-thin probability distribution. By Theorem 6.28, $G - \mathsf{Y}$ admits a vertex-partition $\mathcal{P}_{\mathsf{Y}}$ such that $(G - \mathsf{Y})/\mathcal{P}_{\mathsf{Y}}$ has treewidth at most $t-2$ and each part of $\mathcal{P}_{\mathsf{Y}}$ has at most $c_{6.29}(t) \cdot 2q + 1$ elements. Furthermore, by Theorem 6.27, $\operatorname{ftdfr}_{2q}((G - \mathsf{Y})/\mathcal{P}_{\mathsf{Y}}) \leqslant c_{6.27}(t-2) \cdot (2q)^{t-2}$, and so, there exists a random variable Z' over subsets Z' of $\mathcal{P}_{\mathsf{Y}}$ such that $\operatorname{td}((G - \mathsf{Y})/\mathcal{P}_{\mathsf{Y}} - Z') \leqslant c_{6.27}(t-2) \cdot (2q)^{t-2}$, and Z' is equipped with a $2q$-thin probability distribution. Let

$$\mathsf{Z} = \mathsf{Y} \cup \bigcup \mathsf{Z}'.$$

For every $u \in V(G)$, if $\mathsf{P}_u \in \mathcal{P}_{\mathsf{Y}}$ is such that $u \in \mathsf{P}_u$, then

$$\begin{aligned} \Pr[u \in \mathsf{Z}] &\leqslant \Pr[u \in \mathsf{Y}] + \Pr[\mathsf{P}_u \in \mathsf{Z}'] \\ &\leqslant \frac{1}{2q} + \frac{1}{2q} = \frac{1}{q}. \end{aligned}$$

Therefore, the probability distribution of Z is q-thin. Finally,

$$\operatorname{td}(G - \mathsf{Z}) \leqslant (c_{6.29}(t) \cdot 2q + 1) \cdot (c_{6.27}(t-2) \cdot (2q)^{t-2}) \leqslant c \cdot q^{t-2}.$$

This proves that $\operatorname{ftdfr}_q(G) \leqslant c \cdot q^{t-1}$. □

[5] In other words, there exists a graph A of treewidth at most $t-2$ such that $G \subseteq A \boxtimes K_{w+1}$, where $\boxtimes$ denotes the strong product.

To conclude the proof of Theorem 1.33, it suffices to show Lemma 6.29.

Proof of Lemma 6.29. Let t be a positive integer. We set $c = 2c_{\mathrm{LRS}}(t)$. Let G be a K_t-minor-free graph. By Theorem 6.1, G admits a layered RS-decomposition $(T, \mathcal{W}, \mathcal{A}, \mathcal{D}, \mathcal{L})$ of width at most $c_{\mathrm{LRS}}(t)$. Let q be a positive integer. We root T at an arbitrary vertex s. Let $\mathcal{W} = (T, (W_x \mid x \in V(T)))$, $\mathcal{A} = (A_x \mid x \in V(T))$, $\mathcal{D} = (\mathcal{D}_x \mid x \in V(T))$, and $\mathcal{L} = (\mathcal{L}_x \mid x \in V(T))$, where $\mathcal{L}_x = (L_{x,i} \mid i \in \mathbb{N})$ and $\mathcal{D}_x = (T_x, (D_{x,z} \mid z \in V(T_x)))$ for every $x \in V(T)$. For every vertex $u \in V(G)$, let s_u be the root of the subtree of T induced by $\{x \in V(T) \mid u \in W_x\}$. For every $x \in V(T)$, let $\mathsf{i}(x)$ be a random variable over $\{0, \dots, q-1\}$ with uniform distribution. Then let Y be a the random variable defined by

$$\mathsf{Y} = \{u \in V(G) \mid u \in \textstyle\bigcup_{k \in \mathbb{N}} L_{s_u, \mathsf{i}(s_u)+kq}\}.$$

First, observe that for every $u \in V(G)$,

$$\Pr[u \in \mathsf{Y}] = \begin{cases} 0 & \text{if } u \in A_{s_x}, \\ \frac{1}{q} & \text{otherwise,} \end{cases}$$

and so the distribution of Y is q-thin.

We now bound $\mathrm{tw}(G - \mathsf{Y})$. Let $x \in V(T)$. Let U_x be $W_x \cap W_{\mathrm{p}(T,x)}$ if $x \neq s$, and $\emptyset$ otherwise. Note that $|U_x| \leqslant c_{\mathrm{LRS}}(t)$. For every connected component C of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x - U_x - \mathsf{Y}$, for every $z \in V(T_x)$, $V(C)$ intersects at most $q-1$ layers of $\mathcal{L}_x$ and so

$$|D_{x,z} \cap V(C)| \leqslant c_{\mathrm{LRS}}(t) \cdot (q-1).$$

It follows that $\mathrm{tw}(C) \leqslant c_{\mathrm{LRS}}(t) \cdot (q-1) - 1$ as witnessed by $(T_x, (D_{x,z} \cap V(C) \mid z \in V(T_x)))$. Therefore, $\mathrm{tw}(\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x - U_x - \mathsf{Y}) \leqslant c_{\mathrm{LRS}}(t) \cdot (q-1) - 1$, and so

$$\begin{aligned} \mathrm{tw}(\mathrm{torso}_{G,\mathcal{W}}(W_x) - \mathsf{Y}) &\leqslant |A_x| + |U_x| + c_{\mathrm{LRS}}(T) \cdot (q-1) - 1 \\ &\leqslant c_{\mathrm{LRS}}(t) \cdot (q+1) - 1 \leqslant 2c_{\mathrm{LRS}}(t) \cdot q. \end{aligned}$$

This implies that $\mathrm{tw}(G - \mathsf{Y}) \leqslant c \cdot q$. □

6.7.2 The bounded treewidth case

We now prove the bounds in Theorem 1.36 for graphs of bounded treewidth. See Figures 6.2 and 6.3 for the general plan of the proof.

Lemma 6.30. *Let G be a graph and let $S \subseteq V(G)$. For every path partition $(P_1, \dots, P_\ell)$ of (G, S),*

$$\mathrm{td}(G, S) \leqslant \lceil \log(\ell + 1) \rceil \cdot \max_{i \in [\ell]} |P_i|.$$

Proof. Let $(P_1, \dots, P_\ell)$ be a path partition of (G, S), and let $N = \max_{i \in [\ell]} |P_i|$. It is enough to show that $\mathrm{td}(G, S) \leqslant k \cdot N$ if $\ell = 2^k - 1$. We proceed by induction on k. When $k = 1$ this is clear. Now suppose $k > 1$. Let $S_0 = P_{2^{k-1}}$. Note that S_0 intersects every path in G between $\bigcup_{i \in [2^{k-1}-1]} P_i$ and $\bigcup_{i \in \{2^{k-1}+1, \dots, 2^k\}} P_i$ in G. Thus, there exists a partition V_1, V_2 of $V(G - S_0)$

such that $\bigcup_{i\in[2^{k-1}-1]} P_i \subseteq V_1$, $\bigcup_{i\in\{2^{k-1}+1,\dots,2^k-1\}} P_i \subseteq V_2$, and there is no edge between V_1 and V_2 in G. Then, by the induction hypothesis, Lemma 6.16, and Observation 6.15,

$$\begin{aligned}\operatorname{td}(G,S) &\leqslant \operatorname{td}(G,S_0) + \max\{\operatorname{td}(G[V_1], S\cap V_1), \operatorname{td}(G[V_2], S\cap V_2)\}\\ &\leqslant |S_0| + (k-1)\cdot N\\ &\leqslant k\cdot N.\end{aligned}$$

□

Lemma 6.31. *Let T be a tree. There exists an integer c such that the following holds. Let q be an integer with $q \geqslant 2$, let G be a graph, and let $\mathcal{F}$ be a family of connected subgraphs such that G has no $\mathcal{F}$-rich model of T. There exists $S \subseteq V(G)$ such that*

(a) *$V(F)\cap S \neq \emptyset$ for every $F \in \mathcal{F}$,*

(b) *for every connected component C of $G-S$, $N_G(V(C))$ intersects at most two connected components of $G - V(C)$, and*

(c) *$\operatorname{ftdfr}_q(G,S) \leqslant c\cdot(\operatorname{tw}(G)+1)\cdot\log q$.*

Proof. Let h, d be positive integers such that $T \subseteq F_{h,d}$. Let

$$c = 4(h+1)h(d+h-1).$$

Let $\mathcal{D}$ be a tree decomposition of G of width $\operatorname{tw}(G)$. By Lemma 6.26, there exist pairwise disjoint $S_1,\dots,S_{h+1} \subseteq V(G)$, and for every $a\in[h+1]$ a path partition $(P_{a,0},\dots,P_{a,\ell_a})$ of $(G - S_1 - \dots - S_{a-1}, S_a)$ such that for $S = \bigcup_{a\in[h+1]} S_a$,

6.26.(a) $V(F)\cap S \neq \emptyset$ for every $F\in\mathcal{F}$;

6.26.(b) for every connected component C of $G-S$, $N_G(V(C))$ intersects at most two connected components of $G-S$;

6.26.(c) for every $a\in[h+1]$, for every $i\in[\ell_a]$, $P_{a,i}$ is contained in the union of at most $2h(d+h-1)$ bags of $\mathcal{D}$.

Let q be an integer with $q\geqslant 2$. For every $a\in[h+1]$, let $\mathsf{i}(a)$ be a random variable over $\{0,\dots,q-1\}$ with uniform distribution. Now, let

$$\mathsf{Y} = \bigcup\{P_{a,i} \mid a\in[h+1], i\in\{0,\dots,\ell_a\}, i\equiv \mathsf{i}(a) \bmod q\}.$$

First, observe that for every $u\in S$, $\Pr[u\in\mathsf{Y}] = \frac{1}{q}$, and so the distribution of Y is q-thin. Moreover, for every $a\in[h+1]$, by Lemma 6.30,

$$\begin{aligned}\operatorname{td}(G-S_1-\dots-S_{a-1}-\mathsf{Y}, S_a-\mathsf{Y}) &\leqslant 2h(d+h-1)\cdot(\operatorname{tw}(G)+1)\cdot(\log q+1)\\ &\leqslant 4h(d+h-1)\cdot(\operatorname{tw}(G)+1)\cdot\log q\end{aligned}$$

By Lemma 6.16, it follows that

$$\operatorname{td}(G-\mathsf{Y}, S\setminus\mathsf{Y}) \leqslant 4(h+1)h(d+h-1)\cdot(\operatorname{tw}(G)+1)\cdot\log q = c\cdot(\operatorname{tw}(G)+1)\cdot\log q.$$

This shows that

$$\operatorname{ftdfr}_q(G,S)\leqslant c\cdot(\operatorname{tw}(G)+1)\cdot\log q,$$

which concludes the proof the lemma. □

We will now prove the following main bound on fractional treedepth-fragility rates.

Theorem 6.32. *Let t be an integer with $t \geqslant 2$, and let $X \in \mathcal{S}_t$. There exists an integer c such that for every positive integer q, for every X-minor-free graph G,*

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2}.$$

To prove this theorem, we will apply Theorem 6.2.

Proof of Theorem 6.32. By Lemma 6.18, the family of focused graph parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ defined by

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\,\mathrm{ftdfr}_q(G,S)$$

for every positive integer q, for every graph G, and for every $S \subseteq V(G)$, is nice. Then, we show by induction on t that $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding. Note that, by Lemma 6.21, $\mathcal{S}_{t-1}$ has the coloring elimination property. When $t = 2$, Lemma 1.17 and Lemma 5.2 implies that $q \mapsto 1$ is $(\mathrm{par}, \mathcal{R}_1)$-bounding, and so, by Theorem 6.23, $q \mapsto 1$ is $(\mathrm{par}, \mathcal{S}_2)$-bounding. When $t = 3$, Theorem 6.3 and the induction hypothesis implies that $q \mapsto q$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-bounding. Then, Theorem 6.24 implies that $q \mapsto q$ is $(\mathrm{par}, \mathcal{S}_3)$-bounding. Now suppose $t \geqslant 4$. In particular, $\mathcal{S}_{t-1}$ is closed under leaf addition. Therefore, by the induction hypothesis and Theorem 6.3, $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_{t-1}))$-bounding. Finally, by Theorem 6.4, $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding.

Let $X \in \mathcal{S}_t$ and let $k = |V(X)|$. Since $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_{t-1}))$-bounding, there exists a constant β such that the following holds. Let q be a positive integer, let G be an X-minor-free graph, and let $\mathcal{F}$ be the family of all the one-vertex subgraphs of G. Since G is X-minor-free, there is no $\mathcal{F}$-rich model of X in G. Therefore, there exists $S \subseteq V(G)$ such that, in particular,

(i) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so $S = V(G)$,

(ii) $\mathrm{par}_q(G,S) \leqslant \beta \cdot q^{t-2}$.

We deduce that $\mathrm{ftdfr}_q(G) = \mathrm{ftdfr}_q(G,S) \leqslant \beta \cdot (\mathrm{tw}(G)+2) \cdot q^{t-2} \leqslant 2\beta \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2}$, which proves the theorem for $c = 2\beta$. □

Theorem 6.33. *Let t be an integer with $t \geqslant 2$, and let $X \in \mathcal{R}_t$. There exists an integer c such that for every integer q with $q \geqslant 2$, for every X-minor-free graph G,*

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q.$$

Proof. By Lemma 6.18, the family of focused graph parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ defined by

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\,\mathrm{ftdfr}_q(G,S)$$

for every positive integer q, for every graph G, and for every $S \subseteq V(G)$, is nice. Then, we show by induction on t that $q \mapsto q^{t-2} \log q$ is $(\mathrm{par}, \mathcal{R}_t)$-bounding. When $t = 2$, this is given by Lemma 6.31. Now suppose $t \geqslant 3$. By Lemma 6.21, $\mathcal{R}_{t-1}$ has the coloring elimination property. Therefore, by the induction hypothesis and Theorem 6.3, $q \mapsto q^{t-2} \log q$ is $(\mathrm{par}, \mathbf{A}(\mathcal{R}_{t-1}))$-bounding. Finally, by Theorem 6.4, $q \mapsto q^{t-2} \log q$ is $(\mathrm{par}, \mathcal{R}_t)$-bounding.

Let $X \in \mathcal{R}_t$, and let $k = |V(X)|$. Since $q \mapsto q^{t-2} \log q$ is $(\mathrm{par}, \mathbf{A}(\mathcal{R}_{t-1}))$-bounding, there exists a constant β such that the following holds. Let q be an integer with $q \geqslant 2$, let G be an X-minor-free graph, and let $\mathcal{F}$ be the family of all the one-vertex subgraphs of G. Since G is X-minor-free, there is no $\mathcal{F}$-rich model of X in G. Therefore, there exists $S \subseteq V(G)$ such that, in particular,

(i) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so $S = V(G)$,

(ii) $\mathrm{par}_q(G, S) \leqslant \beta \cdot q^{t-2} \log q$.

We deduce that $\mathrm{ftdfr}_q(G) = \mathrm{ftdfr}_q(G, S) \leqslant \beta \cdot (\mathrm{tw}(G) + 2) \cdot q^{t-2} \log q \leqslant 2\beta \cdot (\mathrm{tw}(G) + 1) \cdot q^{t-2} \log q$, which proves the theorem for $c = 2\beta$. □

6.7.3 The general case

We can now deduce using Lemma 6.29 the bounds in Theorem 1.36 which are independent of the treewidth.

Theorem 6.34. *Let t be an integer with $t \geqslant 2$, and let $X \in \mathcal{S}_t$. There exists an integer c such that for every positive integer q, for every X-minor-free graph G,*

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1}.$$

Proof. Let G be an X-minor-free graph and let q be a positive integer. By Lemma 6.29 applied for $2q$, there exists a random variable Y_1 over the subsets of $V(G)$ such that $\mathrm{tw}(G - \mathsf{Y}_1) \leqslant c_{6.29}(|V(X)|) \cdot q$, and Y_1 is equipped with a $2q$-thin probability distribution. Now, by Theorem 6.32, there exists a positive integer c_0 depending only on X such that

$$\mathrm{ftdfr}_q(G - \mathsf{Y}_1) \leqslant c_0 \cdot (\mathrm{tw}(G - \mathsf{Y}) + 1) \cdot q^{t-2}.$$

Let Y_2 be a random variable over subsets of $V(G - \mathsf{Y}_1)$ witnessing this fact. Then, the random variable $\mathsf{Y} = \mathsf{Y}_1 \cup \mathsf{Y}_2$ is such that

$$\begin{aligned}\mathrm{td}(G - \mathsf{Y}) &\leqslant c_0 \cdot (\mathrm{tw}(G - \mathsf{Y}) + 1) \cdot q^{t-2} \\ &\leqslant c_0 \cdot (c_{6.29}(|V(X)|) \cdot q + 1) \cdot q^{t-2} \leqslant c \cdot q^{t-1}\end{aligned}$$

for $c = c_0 \cdot (c_{6.29}(|V(X)|) + 1)$, and for every $u \in V(G)$, $\Pr[u \in \mathsf{Y}] \leqslant \Pr[u \in \mathsf{Y}_1] + \Pr[u \in \mathsf{Y}_2] \leqslant \frac{1}{2q} + \frac{1}{2q} = \frac{1}{q}$. Therefore, $\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1}$. □

Theorem 6.35. *Let t be a positive integer, and let $X \in \mathcal{R}_t$. There exists an integer c such that for every integer q with $q \geqslant 2$, for every X-minor-free graph G,*

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1} \log q.$$

Proof. Let G be an X-minor-free graph and let q be an integer with $q \geqslant 2$. By Lemma 6.29 applied for $2q$, there exists a random variable Y_1 over the subsets of $V(G)$ such that $\mathrm{tw}(G - \mathsf{Y}_1) \leqslant c_{6.29}(|V(X)|) \cdot q$, and Y_1 is equipped with a $2q$-thin probability distribution. Now, by Theorem 6.33, there exists a positive integer c_0 depending only on X such that

$$\mathrm{ftdfr}_q(G - \mathsf{Y}_1) \leqslant c_0 \cdot (\mathrm{tw}(G - \mathsf{Y}) + 1) \cdot q^{t-2} \log q.$$

Let Y_2 be a random variable over subsets of $V(G - \mathsf{Y}_1)$ witnessing this fact. Then, the random variable $\mathsf{Y} = \mathsf{Y}_1 \cup \mathsf{Y}_2$ is such that

$$\begin{aligned}\mathrm{td}(G - \mathsf{Y}) &\leqslant c_0 \cdot (\mathrm{tw}(G - \mathsf{Y}) + 1) \cdot q^{t-2} \log q \\ &\leqslant c_0 \cdot (c_{6.29}(|V(X)|) \cdot q + 1) \cdot q^{t-2} \log q \leqslant c \cdot q^{t-1} \log q\end{aligned}$$

for $c = c_0 \cdot (c_{6.29}(|V(X)|) + 1)$, and for every $u \in V(G)$, $\Pr[u \in \mathsf{Y}] \leqslant \Pr[u \in \mathsf{Y}_1] + \Pr[u \in \mathsf{Y}_2] \leqslant \frac{1}{2q} + \frac{1}{2q} = \frac{1}{q}$. Therefore, $\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1} \log q$. □

6.8 Weak coloring numbers

In this section, we prove Theorem 1.35. We start with a few observations.

Observation 6.36. *Let G be a graph, $S \subseteq V(G)$, and $U \subseteq V(G)$. For every nonnegative integer q, we have*

$$\mathrm{wcol}_q(G - U, S \setminus U) \leqslant \mathrm{wcol}_q(G, S).$$

Let G be a graph, and let $u, v \in V(G)$. A (u, v)-*geodesic* in G is a path P with endpoints u and v such that P has minimum length among every (u, v)-path. The following lemma is folklore, see e.g. [DHH+24, Lemma 23] for a proof.

Lemma 6.37. *Let G be a graph and let q be a nonnegative integer. For every geodesic Q in G and for every vertex $v \in V(G)$,*

$$|N^q[v] \cap V(Q)| \leqslant 2q + 1.$$

Geodesics are a useful tool when bounding weak coloring numbers. For instance, Lemma 6.37 implies the following.

Observation 6.38. *Let G be a graph, let $S \subseteq V(G)$, let ℓ be a positive integer, and let $Q_1, \dots, Q_\ell$ be geodesics in G. For every nonnegative integer q, we have*

$$\mathrm{wcol}_q(G, S \cup V(Q_1) \cup \dots \cup V(Q_\ell)) \leqslant \mathrm{wcol}_q(G, S) + \ell(2q + 1).$$

We will also use the following bound on weak coloring numbers of paths as a black box.

Lemma 6.39 ([JM22])**.** *For every positive integer q and for every path P, $\mathrm{wcol}_q(P) \leqslant 2 + \lceil \log q \rceil$.*

6.8.1 The bounded treewidth case

In this section, we prove the bounds in Theorem 1.35 that depend on treewidth. See Figures 6.2 and 6.3 for the general plan of the proof. Recall that for all positive integers h and d, we denote by $F_{h,d}$ the (rooted) complete d-ary tree of vertex-height h.

Lemma 6.40. *Let X be a forest. There exists a positive integer $\beta(X)$ such that, for every integer q with $q \geqslant 2$, for every positive integer k, for every graph G with $\mathrm{tw}(G) < k$, for every family $\mathcal{F}$ of connected subgraphs of G, if G has no $\mathcal{F}$-rich model of X, then there is a set $S \subseteq V(G)$ such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;*

(b) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most two connected components of $G - V(C)$;*

(c) *$\mathrm{wcol}_q(G, S) \leqslant \beta(X) \cdot k \cdot \log q$.*

Proof. Since X is a forest, there exist positive integers h, d such that $X \subseteq F_{h,d}$. Let

$$\beta(X) = 6(h + 1)h(d + h - 1).$$

Let k, q be positive integers with $q \geqslant 2$. Let G be a graph of treewidth less than k. By Lemma 5.3, there is a tree decomposition $\mathcal{D}$ of G of width at most $k - 1$ which is natural. Fix such a tree decomposition $\mathcal{D}$. Let $\mathcal{F}$ be a family of connected subgraphs of G. Suppose that G has no $\mathcal{F}$-rich model of X. By Lemma 6.26, there exist pairwise disjoint $S_1, \dots, S_{h+1} \subseteq V(G)$, and for every $a \in [h + 1]$ a path partition $(P_{a,0}, \dots, P_{a,\ell_a})$ of $(G - (S_1 \cup \dots \cup S_{a-1}), S_a)$ such that for $S = \bigcup_{a \in [h+1]} S_a$, 6.26.(a) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;

6.26.(b) for every connected component C of $G - S$, $N_G(V(C))$ intersects at most two connected components of $G - S$;

6.26.(c) for every $a \in [h+1]$, for every $i \in [\ell_a]$, $P_{a,i}$ is contained in the union of at most $2h(d+h-1)$ bags of $\mathcal{D}$.

Note that (a) holds by 6.26.(a), and (b) holds by 6.26.(c). In order to conclude the proof, it suffices to show (c).

Let $a \in [h+1]$. For convenience, let $P_{a,\ell_a+1} = \emptyset$. Consider the path Q with $V(Q) = \{0, \dots, \ell_a + 1\}$ where two numbers are connected by an edge whenever they are consecutive. Let $\sigma' = i_0 \dots i_{\ell_a}$ be an ordering of $\{0, \dots, \ell_a + 1\}$ given by Lemma 6.39, that is such that $\mathrm{wcol}_q(Q, \sigma') \leqslant 1 + \lceil \log q \rceil \leqslant 3 \log q$. For each $i \in \{0, \dots, \ell_a + 1\}$, let σ_i be an arbitrary ordering of P_i. Let σ be the concatenation of $\sigma_{i_0} \dots \sigma_{i_{\ell_a+1}}$ in this order.

Let $u \in V(G) \setminus (S_1 \cup \dots \cup S_{a-1})$, and let $W = \mathrm{WReach}_r[G - (S_1 \cup \dots \cup S_{a-1}), S_a, \sigma, u]$. We argue that

$$|W| \leqslant 6h(d+h-1) \cdot k \cdot \log q.$$

Suppose $W \neq \emptyset$. Let $i_u \in V(T)$ be such that if $u \in S_a$, then $u \in P_{i_u}$, and otherwise, $i_u \in \{0, \dots, \ell\}$ is the least value such that P_{i_u} intersects $N_G(C)$, where C is the connected component of u in $G - S$. Let $A = \mathrm{WReach}_q[Q, \sigma', i_u] \cup \mathrm{WReach}_q[Q, \sigma', i_u + 1]$. In particular, $|A| \leqslant 2 \cdot \mathrm{wcol}_q(Q, \sigma') \leqslant 2 \cdot 3 \log q$. Since $(P_{a,0}, \dots, P_{a,\ell_a+1})$ is a path partition of $(G - (S_1 \cup \dots \cup S_{a-1}), S_a)$, we have $W \subseteq \bigcup_{j \in A} P_j$. By 6.26.(c), for every $j \in \{0, \dots, \ell + 1\}$, P_j is contained in the union of at most d bags of $\mathcal{D}$, and since the width of $\mathcal{D}$ is at most $k - 1$, we have $|P_j| \leqslant 2h(d+h-1)k$. It follows that

$$|W| \leqslant |A| \cdot 2h(d+h-1)k \leqslant 6h(d+h-1)k \log q.$$

This proves that

$$\mathrm{wcol}_q(G - (S_1 \cup \dots \cup S_{a-1}), S_a) \leqslant 6h(d+h-1)k \log q.$$

Now, applying Observation 6.6 inductively, we obtain

$$\begin{aligned}\mathrm{wcol}_q(G, S) &\leqslant \sum_{a \in [h+1]} \mathrm{wcol}_q(G - (S_1 \cup \dots \cup S_{a-1}), S_a) \\ &\leqslant 6(h+1)h(d+h-1)k \log q = \beta(X) \cdot k \cdot \log q.\end{aligned}$$

This shows (c) and concludes the proof of the lemma. □

We are now ready to prove the main results of this section.

Theorem 6.41. *Let t be a positive integer, and let $X \in \mathcal{S}_t$. There exists an integer c such that for every integer q with $q \geqslant 2$, for every X-minor-free graph G,*

$$\mathrm{wcol}_q(G) \leqslant c \cdot (\mathrm{tw}(G) + 1) \cdot q^{t-2}.$$

Proof. For every positive integer q, for every graph G, for every $S \subseteq V(G)$, let

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\,\mathrm{wcol}_q(G,S).$$

By Lemma 6.10, $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ is a nice family of focused parameters.

We will show by induction on t that for every integer t with $t \geqslant 2$, $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding. First consider the case $t = 2$. By Lemma 1.17 and Lemma 5.4, the function $q \mapsto 1$ is $(\mathrm{par}, \mathcal{R}_1)$-bounding. By Theorem 6.23, $q \mapsto 1$ is $(\mathrm{par}, \mathcal{S}_2)$-bounding. Now suppose $t \geqslant 3$ and that $q \mapsto q^{t-3}$ is $(\mathrm{par}, \mathcal{S}_{t-1})$-bounding. If $t = 3$, then by Theorem 6.3, $q \mapsto q$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-bounding, and by Theorem 6.24, $q \mapsto q$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_3))$-bounding. Now suppose $t \geqslant 4$. Recall that $\mathcal{S}_{t-1}$ has the coloring elimination property by Lemma 6.21, and $\mathcal{S}_{t-1}$ is closed under disjoint union. Therefore, by Theorem 6.3, $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_{t-1}))$-bounding. By Theorem 6.4, we deduce that $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding.

Let t be an integer with $t \geqslant 2$, and let $X \in \mathcal{S}_t$. Since $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding, there exists a positive integer β such that the following holds. Let q be a positive integer, let G be an X-minor-free graph, and let $\mathcal{F}$ be the family of all the single vertex subgraphs of G. If G is the null graph, then $\mathrm{wcol}_q(G) = -\infty$. Now suppose G nonnull. Note that there is no $\mathcal{F}$-rich model of X in G. There exists $S \subseteq V(G)$ such that

(i) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so $S = V(G)$, and

(ii) $\mathrm{par}_q(G,S) \leqslant \beta \cdot q^{t-2}$.

Since $\mathrm{wcol}_q(G, V(G)) = \mathrm{wcol}_q(G)$, we deduce that

$$\mathrm{wcol}_q(G) \leqslant (\mathrm{tw}(G)+2) \cdot \mathrm{par}_q(G, V(G)) \leqslant 2\beta \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2},$$

which proves the theorem. □

Theorem 6.42. *Let t be a positive integer, and let $X \in \mathcal{R}_t$. There exists an integer c such that for every integer q with $q \geqslant 2$, for every X-minor-free graph G,*

$$\mathrm{wcol}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q.$$

Proof. For every positive integer q, for every graph G, for every $S \subseteq V(G)$, let

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\,\mathrm{wcol}_q(G,S).$$

By Lemma 6.10, $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ is a nice family of focused parameters.

We will show by induction on t that for every integer t with $t \geqslant 2$, $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}, \mathcal{R}_t)$-bounding. First consider the case $t = 2$. By Lemma 6.40, the function $q \mapsto \log(q+1)$ is $(\mathrm{par}, \mathcal{R}_2)$-bounding. Now suppose $t \geqslant 3$ and that $g \mapsto q^{t-3}\log(q+1)$ is $(\mathrm{par}, \mathcal{R}_{t-1})$-bounding. Recall that $\mathcal{R}_{t-1}$ has the coloring elimination property by Lemma 6.21, and $\mathcal{R}_{t-1}$ is closed under disjoint union. Therefore, by Theorem 6.3, $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}, \mathbf{A}(\mathcal{R}_{t-1}))$-bounding. By Theorem 6.4, we deduce that $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding.

Let t be an integer with $t \geqslant 2$, and let $X \in \mathcal{S}_t$. Since $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding, there exists a positive integer β such that the following holds. Let q be an integer with $q \geqslant 2$, let G be an X-minor-free graph, and let $\mathcal{F}$ be the family of all the single vertex subgraphs of G. Note that there is no $\mathcal{F}$-rich model of X in G. There exists $S \subseteq V(G)$ such that

(i) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so $S = V(G)$, and

(ii) $\mathrm{par}_q(G, S) \leqslant \beta \cdot q^{t-2} \log(q+1) \leqslant 2\beta \cdot q^{t-2} \log q$.

Since $\mathrm{wcol}_q(G, V(G)) = \mathrm{wcol}_q(G)$, we deduce that

$$\mathrm{wcol}_q(G) \leqslant (\mathrm{tw}(G)+2) \cdot \mathrm{par}_q(G, V(G)) \leqslant 4\beta \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q,$$

which proves the theorem. □

6.8.2 The general case

In this section, we prove the bounds in Theorem 1.35 that do not depend on treewidth. The key idea is that we can repeat the same argument as in the bounded treewidth case by replacing Lemma 1.17 by the following lemma.

Lemma 6.43. *For all positive integers k, d, there exists a positive integer $c_{6.43}(k, d)$ such that, for every connected K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, either there are d pairwise vertex-disjoint subgraphs in $\mathcal{F}$, or there exists $S \subseteq V(G)$ such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;*

(b) *$G[S]$ is connected;*

(c) *$\mathrm{wcol}_q(G, S) \leqslant c_{6.43}(k, d) \cdot q$ for every positive integer q.*

Lemma 6.43 is a consequence of the following statement from [DHH+24], which relies on the Graph Minor Structure Theorem by Robertson and Seymour.

Lemma 6.44 ([DHH+24, Lemma 21]). *For all positive integers k, d, there exists a positive integer $c_{6.44}(k, d)$ such that, for every K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G either*

(1) *there are d pairwise vertex-disjoint subgraphs in $\mathcal{F}$, or*

(2) *there exists $A \subseteq V(G)$ with $|A| \leqslant (d-1)c_{6.44}(k)$, and there exists a subgraph X of G which is the union of at most $(d-1)^2 c_{6.44}(k)$ geodesics in $G - A$, such that for every $F \in \mathcal{F}$ we have $V(F) \cap (V(X) \cup A) \neq \emptyset$.*

Proof of Lemma 6.43. Let $c_{6.43}(k, d) = 12(d-1)^2 c_{6.44}(k)$. Let G be a K_k-minor-free graph and let $\mathcal{F}$ be a family of connected subgraphs of G. Suppose that there are no d pairwise disjoint members of $\mathcal{F}$, and hence, Lemma 6.44.(2) holds, yielding $A \subseteq V(G)$ and a subgraph X of G such that $|A| \leqslant (d-1)c_{6.44}(k)$ and X is the union of at most $(d-1)^2 c_{6.44}(k)$ geodesics in $G - A$. Note that $G[A \cup V(X)]$ has at most $|A| + (d-1)^2 c_{6.44}(k)$ connected components. Let $Q_1, \dots, Q_\ell$ be a family of at most $(d-1)c_{6.44}(k) + (d-1)^2 c_{6.44}(k) - 1$ geodesics in G such that the set $S = A \cup V(X) \cup \bigcup_{i \in [\ell]} V(Q_i)$ induces a connected subgraph in G. In particular, $\ell \leqslant 2(d-1)^2 c_{6.44}(k)$. For every positive integer q, by Observations 6.38 and because $\mathrm{wcol}_q(G, A \cup V(X)) \leqslant |A \cup V(X)|$,

$$\begin{aligned}
\mathrm{wcol}_q(G, S) &\leqslant \mathrm{wcol}_q(G, A \cup V(X)) + \ell \cdot (2q+1) \\
&\leqslant |A| + (d-1)^2 c_{6.44}(k) \cdot (2q+1) + \ell \cdot (2q+1) \\
&\leqslant (d-1)c_{6.44}(k) + (d-1)^2 c_{6.44}(k)(2q+1) + 2(d-1)^2 c_{6.44}(k) \cdot (2q+1) \\
&\leqslant 4(d-1)^2 c_{6.44}(k) \cdot (2q+1) \leqslant c_{6.43}(k, d) \cdot q.
\end{aligned}$$

□

Lemma 6.45. *Let k, d be positive integers. Let G be a connected K_k-minor-free graph, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{2,d}$. For every nonempty $U \subseteq V(G)$ such that $G[U]$ is connected, there is a path decomposition $(W_0, \dots, W_\ell)$ of G with $\ell \geqslant 1$ and sets $R_2, \dots, R_\ell \subseteq V(G)$ such that for $S = U \cup \bigcup_{i \in \{2,\dots,\ell\}} (W_{i-1} \cap W_i)$,*

(a) $W_0 = U$;

(b) $V(F) \cap S \neq \emptyset$ *for every* $F \in \mathcal{F}$;

(c) $G[S]$ *is connected;*

(d) $G[R_i]$ *is connected for every* $i \in \{2, \dots, \ell\}$;

(e) $W_{i-1} \cap W_i \subseteq R_i \subseteq \bigcup_{j \in \{0,\dots,i-1\}} W_j$ *for every* $i \in \{2, \dots, \ell\}$;

(f) W_i *and* W_{i+2} *are disjoint for every* $i \in \{0, \dots, \ell - 2\}$; *and*

(g) $\mathrm{wcol}_q(G, R_i) \leqslant (c_{6.43}(k, d+1) + 3) \cdot q$ *for every* $i \in \{2, \dots, \ell\}$ *and for every positive integer* q.

Proof. We proceed by induction on $|V(G)| - |U|$. Let $U \subseteq V(G)$ be nonempty such that $G[U]$ is connected. If $\mathcal{F}|_{G-U} = \emptyset$, then it suffices to take $W_0 = W_1 = U$, $\ell = 1$. In particular, this is the case for $U = V(G)$. Therefore, assume $|U| < |V(G)|$ and $\mathcal{F}|_{G-U} \neq \emptyset$. Let $\mathcal{F}_0$ be the family of all the connected subgraphs A of $G - U$ such that A contains a member of $\mathcal{F}$ and $V(A) \cap N_G(U) \neq \emptyset$. We argue that $\mathcal{F}_0 \neq \emptyset$. Since $\mathcal{F}|_{G-U} \neq \emptyset$, there is a connected component C of $G - U$ containing a member of $\mathcal{F}$. Since G is connected, $V(C) \cap N_G(U) \neq \emptyset$ and so $C \in \mathcal{F}_0$.

Observe that any collection of $d+1$ pairwise disjoint $A_1, \dots, A_{d+1} \in \mathcal{F}_0$ yields an $\mathcal{F}$-rich model of $F_{2,d}$. Indeed, it suffices to take $U \cup A_{d+1}$ as the branch set corresponding to the root of $F_{2,d}$ and $A_1, \dots, A_d$ as the branch sets of the remaining d vertices of $F_{2,d}$. Therefore, there are no $d+1$ pairwise disjoint members of $\mathcal{F}_0$, and thus, by Lemma 6.43 applied to G and $\mathcal{F}_0$, there exists a set $S_0 \subseteq V(G)$ such that

6.43.(a) $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}_0$;

6.43.(b) $G[S_0]$ is connected;

6.43.(c) $\mathrm{wcol}_q(G, S_0) \leqslant c_{6.43}(k, d+1) \cdot q$ for every positive integer q.

Since $\mathcal{F}_0 \neq \emptyset$, we have $S_0 \setminus U \neq \emptyset$. Let Q be a (U, S_0)-geodesic in G (possibly just a one-vertex path), and let $S_1 = S_0 \cup V(Q)$. Note that by 6.43.(b), $G[S_1]$ is connected.

Let $\mathcal{C}_0$ be the family of all the connected components C of $G - U - S_1$ such that $N_G(U) \cap V(C) = \emptyset$. Let $U' = V(G) \setminus \bigcup_{C \in \mathcal{C}_0} V(C)$. Observe that $|U'| > |U|$ since $S_0 \setminus U \neq \emptyset$ and U' contains $U \cup S_0$. Let $\mathcal{F}' = \{F \in \mathcal{F} \mid V(F) \cap U' = \emptyset\}$. By the induction hypothesis applied to G, $\mathcal{F}'$ and U', there is a path decomposition $(W'_0, \dots, W'_{\ell'})$ of G and sets $R'_2, \dots, R'_{\ell'} \subseteq V(G)$ such that for $S' = U' \cup \bigcup_{i \in \{2,\dots,\ell'\}} (W'_{i-1} \cap W'_i)$,

(a') $W'_0 = U'$;

(b') $V(F) \cap S' \neq \emptyset$ for every $F \in \mathcal{F}'$;

(c') $G[S']$ is connected;

(d') $G[R'_i]$ is connected for every $i \in \{2, \ldots, \ell'\}$;

(e') $W'_{i-1} \cap W'_i \subseteq R'_i \subseteq \bigcup_{j \in \{0,\ldots,i-1\}} W'_j$ for every $i \in \{2, \ldots, \ell'\}$;

(f') W'_i and W'_{i+2} are disjoint for every $i \in \{0, \ldots, \ell' - 2\}$; and

(g') $\mathrm{wcol}_q(G, R'_i) \leqslant (c_{6.43}(k, d + 1) + 3) \cdot q$ for every $i \in \{2, \ldots, \ell\}$ and for every positive integer q.

Let $\ell = \ell' + 1$, $W_0 = U$, $W_1 = U'$, $W_2 = (W'_1 \setminus U') \cup (S_1 \setminus U)$, $W_i = W'_{i-1}$ for every $i \in \{3, \ldots, \ell\}$, $R_2 = S_1$, and $R_i = R'_{i-1}$ for every $i \in \{3, \ldots, \ell\}$. Note that (a) holds by construction. We claim that $(W_0, \ldots, W_\ell)$ is a path decomposition of G and (b)-(g) hold, which completes the proof of the lemma.

Let $u \in V(G)$. We claim that $I = \{i \in \{0, \ldots, \ell\} \mid u \in W_i\}$ is an interval. Since $(W'_0, \ldots, W'_{\ell'})$ is a path decomposition of G, $I' = \{i \in \{0, \ldots, \ell'\} \mid u \in W'_i\}$ is an interval. If $u \notin U' = W'_0$, then $I = \{i \in \{2, \ldots, \ell\} \mid u \in W'_{i-1}\} = \{i + 1 \mid i \in I'\}$, which is an interval too. Now suppose that $u \in U'$, and so $0 \in I'$. If $u \notin S_1 \setminus U$, then $u \notin W_2$ and $u \notin W'_i$ for every $i \in \{2, \ldots, \ell'\}$ by (a') and (f'). Hence $I = \{0, 1\}$ if $u \in U$, and $I = \{1\}$ otherwise, which is an interval in both cases. If $u \in S_1 \setminus U$, then $u \notin U = W_0$, and so $I = \{1\} \cup \{i + 1 \mid i \in I' \setminus \{0\}\}$, which is an interval. This proves that I is an interval.

Let uv be an edge of G. We claim that there exists $i \in \{0, \ldots, \ell\}$ such that $u, v \in W_i$. If there exists $i' \in \{2, \ldots, \ell'\}$ such that $u, v \in W'_{i'}$, then $u, v \in W'_{i'} = W_{i'-1}$ and we are done. Now suppose that u and v are not both in W'_i for every $i \in \{2, \ldots, \ell\}$. Since $(W'_0, \ldots, W'_{\ell'})$ is a path decomposition of G, there exists $i' \in \{0, 1\}$ such that $u, v \in W'_{i'}$. If $i' = 0$, then $u, v \in W'_0 = U' = W_1$. Now suppose that u and v are not both in $W'_0 = U'$, and so, in particular, $i' = 1$. Without loss of generality assume that $v \notin U'$. It follows that $v \in W'_1 \setminus U' \subseteq W_2$. Let C be the connected component of v in $G - U - S_1$. Since $v \notin U'$, C belongs to $\mathcal{C}_0$, and so $N_G(V(C)) \cap U = \emptyset$. It follows that $N_G(V(C)) \cap U = \emptyset$, and so, $u \in S_1 \setminus U$. Therefore, $u \in W_2$, which concludes the claim. Furthermore, we obtained that $(W_0, \ldots, W_\ell)$ is a path decomposition of G.

We now prove (b). Consider $F \in \mathcal{F}$. If F intersects U, then $V(F) \cap S \neq \emptyset$ since $U \subseteq S$. If F intersects $S_1 \setminus U$, then F intersects $W_1 \cap W_2 \subseteq S$. Now suppose that F is disjoint from $U \cup S_1$. Let C be the connected component containing F in $G - U - S_1$. Since C is disjoint from $S_0 \subseteq S_1$, by 6.43.(a), C is not a member of $\mathcal{F}_0$. This implies that $N_G(U) \cap V(C) = \emptyset$, and thus, $C \in \mathcal{C}_0$. In particular, F is disjoint from U', and so, $F \in \mathcal{F}'$. By (b'), $V(F) \cap S' \neq \emptyset$, hence, there exists $i \in \{2, \ldots, \ell'\}$ such that $V(F)$ intersects $W'_{i-1} \cap W'_i$. It follows that $W'_{i-1} \cap W'_i = W_i \cap W_{i+1}$ and so $V(F) \cap S \neq \emptyset$. This proves (b).

Let us pause to underline a simple observation that follows directly from the construction, (a'), and (f'):

($\star$) for every $i \in \{3, \ldots, \ell\}$, we have $W_{i-1} \cap W_i = W'_{i-2} \cap W'_{i-1}$.

By (c'), $S' = U' \cup \bigcup_{i \in \{2,\ldots,\ell'\}} (W'_{i-1} \cap W'_i)$ induces a connected subgraph of G. In particular, every connected component of $G[S'] - U'$ has a neighbor in $N_G(V(G) \setminus U') \subseteq S_1$. Since $G[U \cup S_1]$ is connected, it follows that $(S' \setminus U') \cup U \cup S_1$ induces a connected subgraph of G. However, $S = (S' \setminus U') \cup U \cup S_1$ by ($\star$), which yields (c).

For every $i \in \{3, \ldots, \ell\}$, $R_i = R'_{i-1}$ induces a connected subgraph of G by (d'), and $R_2 = S_1$ induces a connected subgraph of G by definition, hence (d) follows.

For the proof of (e), first, observe that by construction, $\bigcup_{j\in\{0,\dots,i-1\}} W_j = \bigcup_{j\in\{0,\dots,i-2\}} W'_j$ for every $i \in \{2,\dots,\ell\}$. In particular, it follows that $R_i = R'_{i-1} \subseteq \bigcup_{j\in\{0,\dots,i-2\}} W'_j = \bigcup_{j\in\{0,\dots,i-1\}} W_j$ for every $i \in \{3,\dots,\ell\}$ by (e'). Moreover, $R_2 = S_1 \subseteq W_1$. It remains to show that $W_{i-1} \cap W_i \subseteq R_i$ for every $i \in \{2,\dots,\ell'\}$. For $i = 2$, $W_1 \cap W_2 = S_1 \setminus U \subseteq R_2$. For $i \in \{3,\dots,\ell\}$, $W_{i-1} \cap W_i = W'_{i-2} \cap W'_{i-1} \subseteq R'_{i-1} = R'_i$ by ($\star$) and (e'). This gives (e).

For every $i \in \{3,\dots,\ell-2\}$, $W_i \cap W_{i+2} = W'_{i-1} \cap W'_{i+1} = \emptyset$ by (f'). Moreover, $W'_3 = W_4$ is disjoint from $S_1 \setminus U \subseteq W'_0$ by (f'). Hence $W_2 \cap W_4 = W'_1 \cap W'_3 = \emptyset$ by (f'). Similarly, $W'_2 = W_3$ is disjoint from $U' = W'_0$ by (f'). Hence $W_1 \cap W_3 = W'_0 \cap W'_2 = \emptyset$. Finally, $W_0 \cap W_2 = \emptyset$ by construction, and so, (f) holds.

It remains to show (g). First, for every $i \in \{3,\dots,\ell\}$, $R_i = R'_{i-1}$ and so $\mathrm{wcol}_q(G, R_i) \leqslant (c_{6.43}(k, d+1) + 3) \cdot q$ for every positive integer q by (g'). Moreover, $R_2 = S_1 = V(Q) \cup S_0$. Hence

$$\mathrm{wcol}_q(G, R_2) \leqslant c_{6.43}(k, d+1) \cdot q + (2q+1) \leqslant (c_{6.43}(k, d+1) + 3) \cdot q$$

for every positive integer q, using 6.43.(c) and Observation 6.38. This shows that (g) holds, which concludes the proof of the lemma. □

Now we are ready to consider graphs with no $\mathcal{F}$-rich model of a fixed star. This part of the argument follows ideas from the proof by [JM22] that $\mathrm{wcol}_q(P) \leqslant 2 + \lceil \log q \rceil$ for every path P and every positive integer q.

Lemma 6.46. *Let k, d be positive integers. For every integer q such that $q \geqslant 2$, for every connected K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if G has no $\mathcal{F}$-rich model of $F_{2,d}$, then there is a set $S \subseteq V(G)$ such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;*

(b) *$G[S]$ is connected;*

(c) *$\mathrm{wcol}_q(G, S) \leqslant 5(c_{6.43}(k, d+1) + 3) \cdot q \log q$.*

Proof. Let q be an integer with $q \geqslant 2$, let G be a connected K_k-minor-free graph, let $\mathcal{F}$ be a family of connected subgraphs of G, and suppose that G has no $\mathcal{F}$-rich model of $F_{2,d}$. Let q be an integer with $q \geqslant 2$. Let U be an arbitrary singleton of a vertex in G. Lemma 6.45 applied to G, $\mathcal{F}$, and U gives a path decomposition $(W_0, \dots, W_\ell)$ and sets $R_2, \dots, R_\ell \subseteq V(G)$ such that for $S' = U \cup \bigcup_{i\in\{2,\dots,\ell\}} (W_{i-1} \cap W_i)$,

6.45.(a) $W_0 = U$;

6.45.(b) $V(F) \cap S' \neq \emptyset$ for every $F \in \mathcal{F}$;

6.45.(c) $G[S']$ is connected;

6.45.(d) $G[R_i]$ is connected for every $i \in \{1,\dots,\ell\}$;

6.45.(e) $W_{i-1} \cap W_i \subseteq R_i \subseteq \bigcup_{j\in\{0,\dots,i-1\}} W_j$ for every $i \in \{2,\dots,\ell\}$;

6.45.(f) W_i and W_{i+2} are disjoint for every $i \in \{0,\dots,\ell-2\}$; and

6.45.(g) $\mathrm{wcol}_q(G, R_i) \leqslant (c_{6.43}(k, d+1) + 3) \cdot q$ for every $i \in \{2,\dots,\ell\}$.

For convenience, we set $R_1 = U$.

Let $s = \lceil \log(q+1) \rceil$. For every $i \in \{0, \ldots, s\}$, let $I_i = \{i \in \{1, \ldots, \ell\} \mid j = 0 \bmod 2^i\}$. We construct recursively families $\{R'_j\}_{j \in \{1,\ldots,\ell\}}$ and $\{S_i\}_{i \in \{0,\ldots,s\}}$ of subsets of $V(G)$ and a family $\{\sigma_j\}_{j \in \{1,\ldots,\ell\}}$ such that σ_j is an ordering of R'_j for every $j \in \{1, \ldots, \ell\}$. For every $j \in I_s$, let

$$R'_j = R_j \setminus \bigcup_{a \in \{0,\ldots,j-2^s-1\}} W_a$$

and let $S_s = \bigcup_{j \in I_s} R'_j$. Let $j \in I_s$. If $j < 2 \cdot 2^s$, then $j = 2^s$ and $R'_j = R_j$, and so by 6.45.(g), $\mathrm{wcol}_q(G, R'_j) \leqslant (c_{6.43}(k, d+1)+3) \cdot q$. Now assume that $j \geqslant 2 \cdot 2^s$. Since $(W_0, \ldots, W_\ell)$ is a path decomposition of G, $W_{j-2^s-1} \cap W_{j-2^s}$ separates $\bigcup_{a \in \{0,\ldots,j-2^s-1\}} W_a$ and $\bigcup_{a \in \{j-2^s,\ldots,\ell\}} W_a$ in G. Since $W_{j-2^s-1} \cap W_{j-2^s} \subseteq R'_{j-2^s}$ (by 6.45.(e)), by Observation 6.5, we obtain

$$\begin{aligned}\mathrm{wcol}_q\left(G - \bigcup_{a \in \{1,\ldots,j-2^s\} \cap I_s} R'_a, R'_j\right) &= \mathrm{wcol}_q\left(G - (W_{j-2^s-1} \cap W_{j-2^s}), R'_j\right) \\ &= \mathrm{wcol}_q\left(G - \bigcup_{a \in \{0,\ldots,j-2^s-1\}} W_a, R'_j\right).\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{wcol}_q\left(G - \bigcup_{a \in \{0,\ldots,j-2^s-1\}} W_a, R'_j\right) &\leqslant \mathrm{wcol}_q(G, R_j) && \text{by Observation 6.36} \\ &\leqslant (c_{6.43}(k, d+1) + 3) \cdot q && \text{by 6.45.(g).}\end{aligned}$$

Let σ_j be an ordering of R'_j such that

$$\mathrm{wcol}_q\left(G - \bigcup_{a \in \{1,\ldots,j-2^s\} \cap I_s} R'_a, R'_j, \sigma_j\right) \leqslant (c_{6.43}(k, d+1) + 3) \cdot q.$$

Next, let $i \in \{0, \ldots, s-1\}$ and assume that S_{i+1} is defined. Now, for every $j \in I_i \setminus I_{i+1}$, let

$$R'_j = \left(R_j \setminus \bigcup_{a \in \{0,\ldots,j-2^i-1\}} W_a\right) \setminus S_{i+1},$$

and let $S_i = \bigcup_{j \in I_i} R'_j$. Note that $S_{i+1} \subseteq S_i$. Also note that for every $j \in I_i$, $W_{j-1} \cap W_j \subseteq R'_j$ by 6.45.(f). Let $j \in I_i \setminus I_{i+1}$. We have $j - 2^i \in I_{i+1}$, and therefore, $W_{j-2^i-1} \cap W_{j-2^i} \subseteq R'_{j-2^i} \subseteq S_{i+1}$. Since $(W_0, \ldots, W_\ell)$ is a path decomposition of G, $W_{j-2^i-1} \cap W_{j-2^i}$ separates $\bigcup_{a \in \{0,\ldots,j-2^i-1\}} W_a$ and $\bigcup_{a \in \{j-2^i,\ldots,\ell\}} W_a$ in G. It follows by Observation 6.5 that

$$\begin{aligned}\mathrm{wcol}_q(G - S_{i+1}, R'_j) &= \mathrm{wcol}_q\left(G - (W_{j-2^i-1} \cap W_{j-2^i}), R'_j\right) \\ &= \mathrm{wcol}_q\left(G - \bigcup_{a \in \{0,\ldots,j-2^i-1\}} W_a, R'_j\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathrm{wcol}_q\left(G - \bigcup_{a \in \{0,\ldots,j-2^i-1\}} W_a, R'_j\right) &\leqslant \mathrm{wcol}_q(G, R_j) && \text{by Observation 6.36} \\ &\leqslant (c_{6.43}(k, d+1) + 3) \cdot q && \text{by 6.45.(g).}\end{aligned}$$

Let σ_j be an ordering of R'_j such that

$$\mathrm{wcol}_q(G - S_{i+1}, R'_j, \sigma_j) \leqslant (c_{6.43}(k, d+1) + 3) \cdot q.$$

We define $S = S_0$. Now, it suffices to show that (a)-(c) hold. Since $S' \subseteq S$, (a) holds by 6.45.(b).

Recall that $G[S']$ is connected by 6.45.(c). Next, let C be a connected component of $G[R'_j]$ for some fixed $j \in I_s$. If $V(C) \cap (W_{j-1} \cap W_j) \neq \emptyset$, then $V(C) \cap S' \neq \emptyset$, and so, $G[S' \cup V(C)]$ is connected. Thus, assume that $V(C) \cap (W_{j-1} \cap W_j) = \emptyset$. However, $W_{j-1} \cap W_j \subseteq R'_j$, hence, C has a neighbor in W_{j-2^s-1}, in particular, in $W_{j-2^s-1} \cap W_{j-2^s} \subseteq S'$. Hence, again $G[S' \cup V(C)]$ is connected. In particular, we have just proved that $G[S' \cup S_s]$ is connected. Next, suppose that $G[S' \cup S_{i+1}]$ is connected for some $i \in \{0, \dots, s-1\}$. Let C be a connected component of $G[R'_j]$ for some fixed $j \in I_i$. If $V(C) \cap (W_{j-1} \cap W_j) \neq \emptyset$, then $V(C) \cap S' \neq \emptyset$, and so, $G[S' \cup V(C)]$ is connected. Thus, assume that $V(C) \cap (W_{j-1} \cap W_j) = \emptyset$. However, $W_{j-1} \cap W_j \subseteq R'_j$, hence, C has a neighbor in $W_{j-2^i-1} \cup S_{i+1}$, in particular, in $(W_{j-2^i-1} \cap W_{j-2^i}) \cup S_{i+1} \subseteq S' \cup S_{i+1}$. Hence, $G[S' \cup S_{i+1} \cup V(C)]$ is connected. Finally, $G[S' \cup S_0] = G[S]$ is connected, which yields (b).

The sets $\{R'_j\}_{j \in \{2,\dots,\ell\}}$ are pairwise disjoint, and they partition S. Let σ be an ordering of S such that

(i) σ extends σ_j, for every $j \in \{1, \dots, \ell\}$;

(ii) for every $j, j' \in I_s$ with $j < j'$, for all $u \in R'_j$ and $v \in R'_{j'}$, $u <_\sigma v$; and

(iii) for every $i \in \{0, \dots, s-1\}$, for all $u \in S_{i+1}$ and $v \in S_i \setminus S_{i+1}$, $u <_\sigma v$.

For convenience, let $R'_0 = \emptyset$ and $W_j = R'_j = \emptyset$ for every integer j with $j > \ell$.

We now show (c). Let $u \in V(G)$. We will show that $|\operatorname{WReach}_q[G, S, \sigma, u]| \leqslant 5(c_{6.43}(k, d+1) + 3) \cdot q \log q$. Let $j_u \in \{0, \dots, \ell\}$ be minimum such that $u \in W_{j_u}$. We claim that

$$|\operatorname{WReach}_q[G, S, \sigma, u] \cap S_s| \leqslant 2(c_{6.43}(k, d+1) + 3) \cdot q.$$

Let $\alpha = \max\{0\} \cup \{a \in I_s \mid a \leqslant j_u\}$, and let $\beta = \alpha + 2^s$. Thus, if $\beta \leqslant \ell$, then $\beta \in I_s$. Next, we argue that

$$\operatorname{WReach}_q[G, S, \sigma, u] \cap S_s \subseteq R'_\alpha \cup R'_\beta.$$

Suppose to the contrary that there is a vertex $v \in \operatorname{WReach}_q[G, S, \sigma, u] \cap S_s$ with $v \notin R'_\alpha \cup R'_\beta$. Let $\gamma \in I_s \setminus \{\alpha, \beta\}$ be such that $v \in R'_\gamma$. Then either $\gamma < \alpha$, or $\gamma > \beta$. First assume that $\gamma < \alpha$. Since $R'_\gamma \subseteq \bigcup_{a \in \{0,\dots,\gamma-1\}} W_a$ and because $(W_0, \dots, W_\ell)$ is a path decomposition of G, every (u, v)-path in G intersects $W_{a-1} \cap W_a$ for each $a \in \{\gamma, \dots, j_u\}$. Since $(W_{a-1} \cap W_a)_{a \in \{1,\dots,\ell\}}$ are pairwise disjoint, we deduce that $\operatorname{dist}_G(u, v) \geqslant j_u - \gamma \geqslant \alpha - \gamma \geqslant 2^s > q$, which contradicts the fact that $v \in \operatorname{WReach}_r[G, S, \sigma, u]$. Finally, assume $\gamma > \beta$. Note that $\gamma \leqslant \ell$ since $R'_\gamma \neq \emptyset$ as $v \in R'_\gamma$. Since $R'_\gamma \subseteq \bigcup_{a \in \{\gamma-2^s,\dots,\gamma-1\}} W_a$, and because $(W_0, \dots, W_\ell)$ is a path decomposition of G, every (u, v)-path in G intersects $W_{\beta-1} \cap W_\beta$. However, for every $w \in W_{\beta-1} \cap W_\beta$, we have $w <_\sigma v$, thus, $v \notin \operatorname{WReach}_q[G, S, \sigma, u]$, which is a contradiction. We obtain that $\operatorname{WReach}_q[G, S, \sigma, u] \cap S_s \subseteq R'_\alpha \cup R'_\beta$.

For every $\varepsilon \in \{\alpha, \beta\}$, by definition of σ, we have

$$\begin{aligned}\operatorname{WReach}_q[G, S, \sigma, u] \cap R'_\varepsilon &\subseteq \operatorname{WReach}_q\left[G - \textstyle\bigcup_{a \in I_s \cap \{1,\dots,\alpha-1\}} R'_a, R'_\varepsilon, \sigma_\varepsilon, u\right] \\ &\subseteq \operatorname{WReach}_q\left[G - \textstyle\bigcup_{a \in \{1,\dots,j_u-2^s\} \cap I_s} R'_a, R'_\varepsilon, \sigma_\varepsilon, u\right]\end{aligned}$$

and therefore,

$$|\operatorname{WReach}_q[G, S, \sigma, u] \cap R'_\varepsilon| \leqslant (c_{6.43}(k, d+1) + 3) \cdot q.$$

In particular,

$$\begin{aligned}|\operatorname{WReach}_q[G,S,\sigma,u]\cap S_s| &\leqslant |\operatorname{WReach}_q[G,S,\sigma,u]\cap(R'_\alpha\cup R'_\beta)|\\ &\leqslant 2(c_{6.43}(k,d+1)+3)\cdot q.\end{aligned}$$

Next, let $i\in\{0,\dots,s-1\}$. We claim that

$$|\operatorname{WReach}_q[G,S,\sigma,u]\cap(S_i\setminus S_{i+1})|\leqslant(c_{6.43}(k,d+1)+3)\cdot q.$$

Since each vertex of S_{i+1} precedes each vertex of S_i in σ, we have

$$\operatorname{WReach}_q[G,S,\sigma,u]\cap(S_i\setminus S_{i+1})\subseteq\operatorname{WReach}_q[G-S_{i+1},S-S_{i+1},\sigma|_{S\setminus S_{i+1}},u]\cap(S_i\setminus S_{i+1}).$$

Let $\alpha=\max\{a\in I_{i+1}\mid a\leqslant j_u\}$ and $\beta=\alpha+2^{i+1}$. Let C be the connected component of u in $G-S_{i+1}$. Since $W_{\alpha-1}\cap W_\alpha, W_{\beta-1}\cap W_\beta\subseteq S_{i+1}$, and because $(W_0,\dots,W_\ell)$ is a path decomposition of G, $V(C)\cap S\subseteq\bigcup_{a\in\{\alpha,\dots,\beta-1\}}W_a$. We deduce that

$$\operatorname{WReach}_q[G-S_{i+1},S\setminus S_{i+1},\sigma|_{S\setminus S_{i+1}},u]\cap(S_i\setminus S_{i+1})\subseteq\textstyle\bigcup_{a\in\{\alpha,\dots,\beta-1\}}W_a.$$

Since the only members of $I_i\setminus I_{i+1}$ in $\{\alpha+1,\dots,\beta-1\}$ is $\gamma=\alpha+2^i$, we in fact have

$$\operatorname{WReach}_q[G-S_{i+1},S\setminus S_{i+1},\sigma|_{S\setminus S_{i+1}},u]\cap(S_i\setminus S_{i+1})\subseteq R'_\gamma,$$

and we deduce that

$$\begin{aligned}|\operatorname{WReach}_q[G-S_{i+1},S\setminus S_{i+1},\sigma|_{S\setminus S_{i+1}},u]\cap(S_i\setminus S_{i+1})| &\leqslant \operatorname{wcol}_q(G-S_{i+1},R'_\gamma,\sigma_\gamma)\\ &\leqslant(c_{6.43}(k,d+1)+3)\cdot q.\end{aligned}$$

For convenience let $S_{s+1}=\emptyset$. Since $S=S_0$, it follows that

$$\begin{aligned}|\operatorname{WReach}_q[G,S,\sigma,u]| &\leqslant\sum_{i\in\{0,\dots,s\}}|\operatorname{WReach}_q[G,S,\sigma,u]\cap(S_i\setminus S_{i+1})|\\ &\leqslant(s+2)\cdot(c_{6.43}(k,d+1)+3)\cdot q\\ &\leqslant 5(c_{6.43}(k,d+1)+3)\cdot q\log q.\end{aligned}$$

□

Lemma 6.47. *Let k,h,d be positive integers with $h\geqslant 2$. There is an integer $c_{6.47}(h,d,k)$ such that for every integer q with $q\geqslant 2$, for every connected K_k-minor-free graph G, for every family $\mathcal{F}$ of connected subgraphs of G, if G has no $\mathcal{F}$-rich model of $F_{h,d}$, then there is a set $S\subseteq V(G)$ such that*

(a) *$V(F)\cap S\neq\emptyset$ for every $F\in\mathcal{F}$;*

(b) *$G[S]$ is connected;*

(c) *$\operatorname{wcol}_q(G,S)\leqslant c_{6.47}(h,d,k)\cdot q\log q$.*

Proof. We proceed by induction on h. For $h=2$, the result is given by Lemma 6.46 setting $c_{6.47}(1,d,k)=5(c_{6.43}(k,d+1)+3)$. Next, assume $h>2$ and that $c_{6.47}(h-1,d,k)$ witnesses the assertion for $h-1$. Let $c_{6.47}(h,d,k)=5(c_{6.43}(k,d+1)+3)+3+c_{6.47}(h-1,d+1,k)$.

Let q be an integer with $q \geqslant 2$, let G be a connected K_k-minor-free graph, and let $\mathcal{F}$ be a family of connected subgraphs of G. Suppose that G has no $\mathcal{F}$-rich model of $F_{h,d}$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of G such that H contains an $\mathcal{F}|_H$-rich model of $F_{h-1,d+1}$. We claim that there is no $\mathcal{F}'$-rich model of $F_{2,d}$ in G. Suppose to the contrary that $(B_x \mid x \in V(F_{2,d}))$ is such a model. Let s be the root of $F_{2,d}$ and let s' be the root of $F_{h-1,d}$. For every $x \in V(F_{2,d}) \setminus \{s\}$, by Lemma 3.16, there is an $\mathcal{F}$-rich model $(C_y \mid y \in V(F_{h-1,d}))$ of $F_{h-1,d}$ in $G[B_x]$ such that $C_{s'}$ contains a vertex of $N_G(B_s) \cap B_x$. The union of these models together with B_s yields an $\mathcal{F}$-rich model of $F_{h,d}$ in G, which is a contradiction. See Figure 6.11.

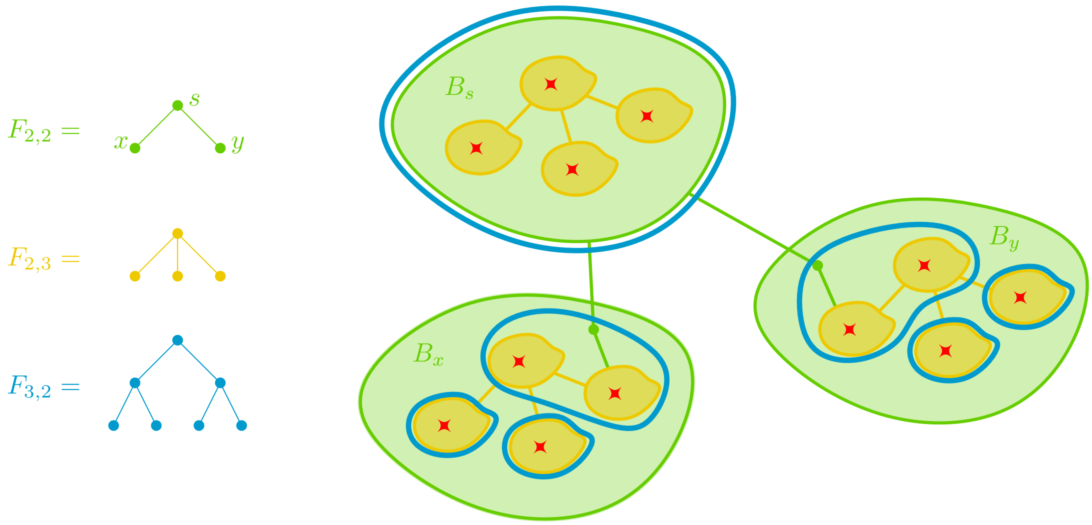


Figure 6.11: We provide an example of the construction of an $\mathcal{F}$-rich model of $F_{h,d}$ in G assuming that there is an $\mathcal{F}'$-rich model of $F_{2,d}$ in G in the case where $h = 3$ and $d = 2$. In green, we depict an $\mathcal{F}'$-rich model of $F_{2,d} = F_{2,2}$ in the graph. Each branch set contains an $\mathcal{F}$-rich model of $F_{h-1,d+1} = F_{2,3}$. We depict these models in yellow and the red stars are the elements of $\mathcal{F}$. The obtained model of $F_{h,d} = F_{3,2}$ we depict in blue. Note that this model is $\mathcal{F}$-rich.

Since G has no $\mathcal{F}'$-rich model of $F_{2,d}$, by Lemma 6.46, there is a set $S_0 \subseteq V(G)$ such that

6.46.(a) for every $F \in \mathcal{F}'$, $V(F) \cap S_0 \neq \emptyset$;

6.46.(b) $G[S_0]$ is connected;

6.46.(c) $\operatorname{wcol}_q(G, S_0) \leqslant 5(c_{6.43}(k, d+1) + 3) \cdot q \log q$.

Let C be a connected component of $G - S_0$. By 6.46.(a), $C \notin \mathcal{F}'$, and so, C has no $\mathcal{F}|_C$-rich model of $F_{h-1,d+1}$. Therefore, by the induction hypothesis, there is a set $S_C \subseteq V(C)$ such that

(a') $V(F) \cap S_C \neq \emptyset$ for every $F \in \mathcal{F}|_C$;

(b') $C[S_C]$ is connected;

(c') $\operatorname{wcol}_q(C, S_C) \leqslant c_{6.47}(h-1, d+1, k) \cdot q \log q$.

Let Q_C be an $(S_C, N_G(S_0))$-geodesic in G. In particular, Q_C is a geodesic in C. Let $\mathcal{C}$ be the family of the connected components of $G - S_0$ and let

$$S = S_0 \cup \bigcup_{C \in \mathcal{C}} (S_C \cup V(Q_C)).$$

See Figure 6.12 for an illustration. We claim that (a)-(c) hold.

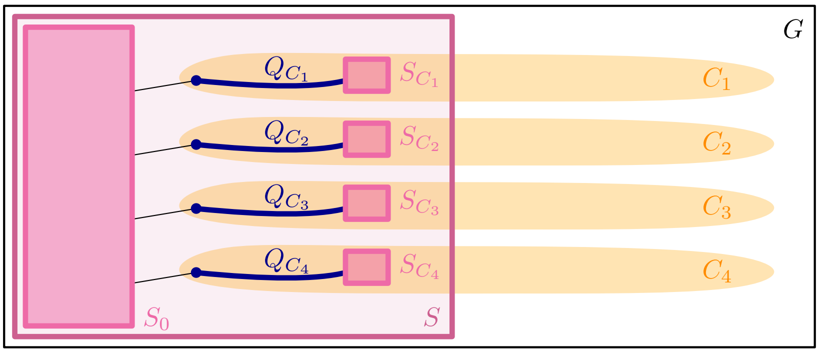


Figure 6.12: An illustration of the construction of the set S in the proof of Lemma 6.47.

Let $F \in \mathcal{F}$. If $V(F) \cap S_0 = \emptyset$, then $V(F) \subseteq V(C)$ for some connected component C of $G - S_0$. In particular, $F \in \mathcal{F}|_C$, and thus, by (a'), $V(F) \cap S_C \neq \emptyset$, which proves (a). The graph $G[S]$ is connected by construction, (b') and 6.46.(b), which yields (b). The following sequence of inequalities concludes the proof of (c) and the lemma:

$$\begin{aligned}
\mathrm{wcol}_q(G,S) &\leqslant \mathrm{wcol}_q(G,S_0) + \mathrm{wcol}_q\left(G - S_0, \textstyle\bigcup_{C\in\mathcal{C}}\left(S_C \cup V(Q_C)\right)\right) && \text{by Observation 6.6}\\
&\leqslant \mathrm{wcol}_q(G,S_0) + \max_{C\in\mathcal{C}} \mathrm{wcol}_q(C, S_C \cup V(Q_C)) && \text{by Observation 6.5}\\
&\leqslant \mathrm{wcol}_q(G,S_0) + \max_{C\in\mathcal{C}} \mathrm{wcol}_q(C, S_C) + (2q+1) && \text{by Observation 6.38}\\
&\leqslant 5(c_{6.43}(k,d+1)+3)\cdot q\log q\\
&\qquad + c_{6.47}(h-1,d+1,k)\cdot q\log q + 3q && \text{by 6.46.(c) and (c')}\\
&\leqslant (5(c_{6.43}(k,d+1)+3)\\
&\qquad + c_{6.47}(h-1,d+1,k)+3)\cdot q\log q\\
&= c_{6.47}(h,d,k)\cdot q\log q.
\end{aligned}$$

□

We are now ready to prove the main results of this section.

Theorem 6.48. *Let t be a positive integer, and let $X \in \mathcal{S}_t$. There exists an integer c such that for every integer q with $q \geqslant 2$, for every X-minor-free graph G,*

$$\mathrm{wcol}_q(G) \leqslant c \cdot q^{t-1}.$$

Proof. By Lemma 6.9, $(\mathrm{wcol}_q \mid q \in \mathbb{N}_{>0})$ is a nice family of focused parameters.

We will show by induction on t that for every integer t with $t \geqslant 2$, $q \mapsto q^{t-1}$ is $(\mathrm{wcol}, \mathcal{S}_t)$-bounding. First consider the case $t = 2$. By Lemma 6.43, the function $q \mapsto q$ is $(\mathrm{wcol}, \mathcal{R}_1)$-bounding. By Theorem 6.23, $q \mapsto q$ is $(\mathrm{wcol}, \mathcal{S}_2)$-bounding. Now suppose $t \geqslant 3$ and that

$g \mapsto q^{t-2}$ is $(\mathrm{wcol}, \mathcal{S}_{t-1})$-bounding. Recall that $\mathcal{S}_{t-1}$ has the coloring elimination property by Lemma 6.21, and $\mathcal{S}_{t-1}$ is closed under disjoint union. Therefore, by Theorem 6.3, $q \mapsto q^{t-1}$ is $(\mathrm{wcol}, \mathbf{A}(\mathcal{S}_{t-1}))$-bounding. By Theorem 6.4, we deduce that $q \mapsto q^{t-1}$ is $(\mathrm{wcol}, \mathcal{S}_t)$-bounding.

Let t be an integer with $t \geqslant 2$, and let $X \in \mathcal{S}_t$. Since $q \mapsto q^{t-2}$ is $(\mathrm{wcol}, \mathcal{S}_t)$-bounding, there exists a positive integer β such that the following holds. Let q be a positive integer, let G be an X-minor-free graph, and let $\mathcal{F}$ be the family of all the single vertex subgraphs of G. Note that there is no $\mathcal{F}$-rich model of X in G. There exists $S \subseteq V(G)$ such that

(i) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so $S = V(G)$, and

(ii) $\mathrm{wcol}_q(G, S) \leqslant \beta \cdot q^{t-1}$.

Since $\mathrm{wcol}_q(G, V(G)) = \mathrm{wcol}_q(G)$, we deduce that

$$\mathrm{wcol}_q(G) \leqslant \beta \cdot q^{t-1},$$

which proves the theorem. □

Theorem 6.49. *Let t be a positive integer, and let $X \in \mathcal{R}_t$. There exists an integer c such that for every integer q with $q \geqslant 2$, for every X-minor-free graph G,*

$$\mathrm{wcol}_q(G) \leqslant c \cdot q^{t-1} \log q.$$

Proof. By Lemma 6.9, $(\mathrm{wcol}_q \mid q \in \mathbb{N}_{>0})$ is a nice family of focused parameters.

We will show by induction on t that for every integer t with $t \geqslant 2$, $q \mapsto q^{t-1} \log(q+1)$ is $(\mathrm{wcol}, \mathcal{R}_t)$-bounding. First consider the case $t = 2$. By Lemma 6.47, the function $q \mapsto q \log(q{+}1)$ is $(\mathrm{wcol}, \mathcal{R}_2)$-bounding. Now suppose $t \geqslant 3$ and that $g \mapsto q^{t-2} \log(q+1)$ is $(\mathrm{wcol}, \mathcal{R}_{t-1})$-bounding. Recall that $\mathcal{R}_{t-1}$ has the coloring elimination property by Lemma 6.21, and $\mathcal{R}_{t-1}$ is closed under disjoint union. Therefore, by Theorem 6.3, $q \mapsto q^{t-1} \log(q{+}1)$ is $(\mathrm{wcol}, \mathbf{A}(\mathcal{R}_{t-1}))$-bounding. By Theorem 6.4, we deduce that $q \mapsto q^{t-1} \log(q+1)$ is $(\mathrm{wcol}, \mathcal{S}_t)$-bounding.

Let t be an integer with $t \geqslant 2$, and let $X \in \mathcal{S}_t$. Since $q \mapsto q^{t-1} \log(q+1)$ is $(\mathrm{wcol}, \mathcal{S}_t)$-bounding, there exists a positive integer β such that the following holds. Let q be an integer with $q \geqslant 2$, let G be an X-minor-free graph, and let $\mathcal{F}$ be the family of all the single vertex subgraphs of G. Note that there is no $\mathcal{F}$-rich model of X in G. There exists $S \subseteq V(G)$ such that

(i) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so $S = V(G)$, and

(ii) $\mathrm{wcol}_q(G, S) \leqslant \beta \cdot q^{t-1} \log(q+1) \leqslant 2\beta \cdot q^{t-2} \log q$.

Since $\mathrm{wcol}_q(G, V(G)) = \mathrm{wcol}_q(G)$, we deduce that

$$\mathrm{wcol}_q(G) \leqslant 4\beta \cdot q^{t-1} \log q,$$

which proves the theorem. □

6.9 Centered colorings

In this section, we prove our upper bounds for centered chromatic numbers, that is Theorem 1.34. The case of K_t-minor-free graphs is covered in Sections 6.9.1 to 6.9.3.

6.9.1 Preliminaries

We will need the following simple combinatorial fact. Let T be a tree and let $\mathcal{Q}$ be a collection of connected subgraphs of T whose vertex sets partition $V(T)$. For every $X \subseteq V(T)$, let $\mathcal{Q}(X) = \{Q \in \mathcal{Q} \mid V(Q) \cap X \neq \emptyset\}$.

Lemma 6.50. *Let T be a tree and let $\mathcal{Q}$ a collection of connected subgraphs of T whose vertex sets partition $V(T)$. Let $X, Y \subseteq V(T)$ with $X \subseteq Y$ and $\mathrm{LCA}(T, X) = X$. If $\mathcal{Q}(X) = \mathcal{Q}(Y)$, then $\mathcal{Q}(Y) = \mathcal{Q}(\mathrm{LCA}(T, Y))$.*

Proof. Suppose that $\mathcal{Q}(X) = \mathcal{Q}(Y)$. Since $Y \subseteq \mathrm{LCA}(T, Y)$, we have $\mathcal{Q}(Y) \subseteq \mathcal{Q}(\mathrm{LCA}(T, Y))$. Thus, it suffices to prove that $\mathcal{Q}(\mathrm{LCA}(T, Y)) \subseteq \mathcal{Q}(Y)$. Consider $Q \in \mathcal{Q}(\mathrm{LCA}(T, Y))$. There exist $y_1, y_2 \in Y$ such that $\mathrm{lca}(T, y_1, y_2) \in Q$. Let $Q_1, Q_2 \in \mathcal{Q}$ be such that $y_1 \in V(Q_1)$, $y_2 \in V(Q_2)$. In particular, $Q_1, Q_2 \in \mathcal{Q}(Y) = \mathcal{Q}(X)$. Hence, there exist $x_1 \in V(Q_1) \cap X$ and $x_2 \in V(Q_2) \cap X$.

If the roots of Q_1 and Q_2 are not in an ancestor-descendant relation in T, then for all $q_1 \in V(Q_1)$ and $q_2 \in V(Q_2)$, we have $\mathrm{lca}(T, q_1, q_2) = \mathrm{lca}(T, \mathrm{root}(Q_1), \mathrm{root}(Q_2))$. In particular,

$$\mathrm{lca}(T, x_1, x_2) = \mathrm{lca}(T, \mathrm{root}(Q_1), \mathrm{root}(Q_2)) = \mathrm{lca}(T, y_1, y_2) \in V(Q).$$

Observe that $\mathrm{lca}(T, x_1, x_2) \in \mathrm{LCA}(T, X) = X$, thus, $V(Q) \cap X \neq \emptyset$, and so, $Q \in \mathcal{Q}(X) = \mathcal{Q}(Y)$.

Therefore, we can assume that one of the roots say $\mathrm{root}(Q_1)$, is an ancestor of the other $\mathrm{root}(Q_2)$ in S. It follows that for all $q_1 \in V(Q_1)$ and $q_2 \in V(Q_2)$, $\mathrm{lca}(T, q_1, q_2)$ is either equal to $\mathrm{root}(Q_1)$ or is a descendant of $\mathrm{root}(Q_1)$. In both cases, $\mathrm{lca}(T, q_1, q_2)$ lies in the path from q_1 to $\mathrm{root}(Q_1)$ in S. Since Q_1 is connected, we obtain that $\mathrm{lca}(T, q_1, q_2) \in V(Q_1)$. In particular, $\mathrm{lca}(T, x_1, x_2) \in V(Q_1)$, and so, $Q_1 = Q$ implying $Q \in \mathcal{Q}(Y)$, which ends the proof. □

Finally, we need the following simple fact about torsos. We give a proof for completeness.

Lemma 6.51. *Let G be a graph, let $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ be a tree decomposition of G, and let $x \in V(T)$. If H is a connected subgraph of G intersecting W_x, then $V(H) \cap W_x$ induces a connected subgraph of $\mathrm{torso}_{G,\mathcal{W}}(W_x)$.*

Proof. Let H be a connected subgraph of G intersecting W_x, and suppose to the contrary that the subgraph H' of $\mathrm{torso}_{G,\mathcal{W}}(W_x)$ induced by $V(H) \cap W_x$ is not connected. Let $u, v \in V(H')$ be vertices in distinct components of H such that the distance between u and v is minimal in H. Note that the internal vertices of a shortest path between u and v in H do not lie in W_x as otherwise we obtain a pair of vertices of H' in distinct components that are closer in H. By the properties of tree decompositions, such a shortest path has all its vertices in $\bigcup_{z \in V(T_{y|x})} W_z$ for some $y \in N_T(x)$, and so $u, v \in W_x \cap W_y$. It follows that u and v are adjacent in $\mathrm{torso}_{G,\mathcal{W}}(W_x)$, and so, in H', which contradicts the assumption and completes the proof. □

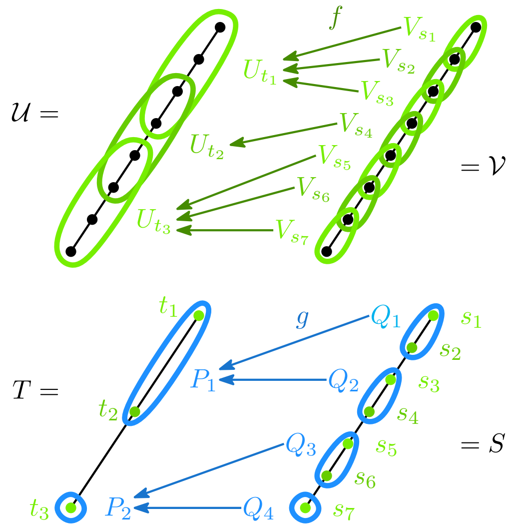


Figure 6.13: In this example, G is an 8-vertex path. $\mathcal{P} = \{P_1, P_2\}$ and $\mathcal{Q} = \{Q_1, Q_2, Q_3, Q_4\}$. One can check that $(\mathcal{V}, \mathcal{Q})$ refines $(\mathcal{U}, \mathcal{P})$ which is witnessed by f and g. Note that f is a function on $V(S)$ but for the readability reasons we depict it as it acted on bags of $\mathcal{V}$.

We will also need a more explicit version of Lemma 5.3. Indeed, at one point in the proof, we would like to assume that a tree decomposition that we consider is natural but still preserves certain properties. For this reason, we introduce the following definitions and we prove Lemma 6.52.

Let G be a graph. A pair $(\mathcal{U}, \mathcal{P})$ is a *normal pair* of G if $\mathcal{U} = (T, (U_t \mid t \in V(T)))$ is a tree decomposition of G, and $\mathcal{P}$ is a collection of connected subgraphs of T whose vertex sets partition $V(T)$. Consider two normal pairs $(\mathcal{U}, \mathcal{P})$ and $(\mathcal{V}, \mathcal{Q})$ of G with $\mathcal{U} = (T, (U_t \mid t \in V(T)))$ and $\mathcal{V} = (S, (V_s \mid s \in V(S)))$. We say that $(\mathcal{V}, \mathcal{Q})$ *refines*, or is a *refinement* of $(\mathcal{U}, \mathcal{P})$ if there exist $f\colon V(S) \to V(T)$ and $g\colon \mathcal{Q} \to \mathcal{P}$ such that

(r1) for every $s \in V(S)$,
$$V_s \subseteq U_{f(s)};$$

(r2) for every $Q \in \mathcal{Q}$,
$$f(V(Q)) \subseteq V(g(Q)).$$

See an example in Figure 6.13.

Lemma 6.52. *Let G be a connected graph and let $(\mathcal{U}, \mathcal{P})$ be a normal pair of G. There exists $(\mathcal{V}, \mathcal{Q})$ a refinement of $(\mathcal{U}, \mathcal{P})$ such that $\mathcal{V}$ is natural.*

Proof. For every positive integer i, for every tree decomposition $\mathcal{W}$ of G, we denote by $n_i(\mathcal{W})$ the number of bags of $\mathcal{W}$ of size i. Then let $n(\mathcal{W}) = (n_{|V(G)|}(\mathcal{W}), \ldots, n_0(\mathcal{W}))$.

Since the refinement relation is reflexive, every normal pair of G has a refinement. Consider the refinement $(\mathcal{V}, \mathcal{Q})$ of $(\mathcal{U}, \mathcal{P})$ with $n(\mathcal{V})$ minimal in the lexicographic order. We claim that $\mathcal{V}$ is natural.

Let $\mathcal{V} = (S, (V_s \mid s \in V(S)))$. Suppose to the contrary that there exist $x, y \in V(S)$ such that $xy \in E(S)$ and $G\left[\bigcup_{s \in V(S_{x|y})} V_s\right]$ is not connected, and let $\mathcal{C}$ be the family of its connected components.

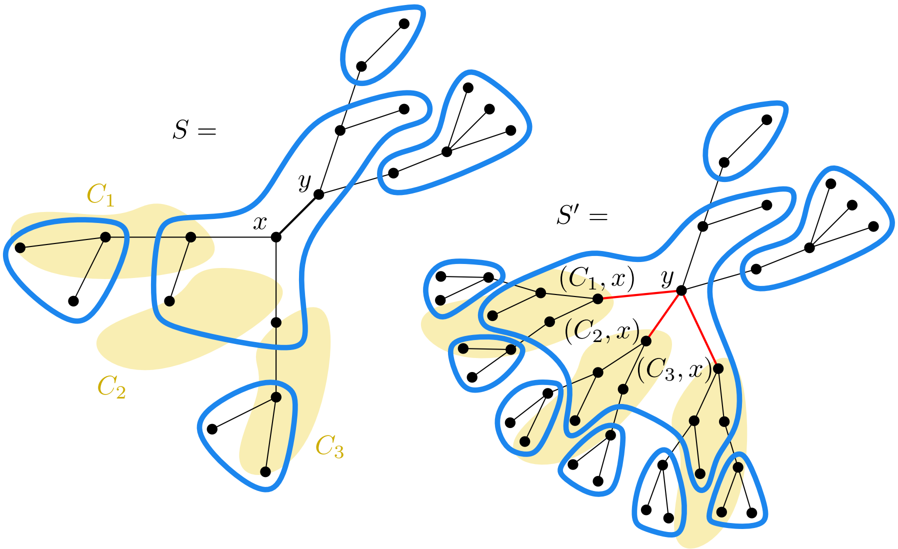


Figure 6.14: Proof of Lemma 6.52: construction of S' from S and of $\mathcal{Q}'$ from $\mathcal{Q}$. The blue partition of S is $\mathcal{Q}$ and the blue partition of S' is $\mathcal{Q}'$.

The plan for obtaining a contradiction is finding a refinement $(\mathcal{V}', \mathcal{Q}')$ of $(\mathcal{V}, \mathcal{Q})$ such that $n(\mathcal{V}') < n(\mathcal{V})$ in the lexicographic order. For every $C \in \mathcal{C}$, let S_C be a copy of $S_{x|y}$ with vertex set $\{(C, s) \mid s \in V(S_{x|y})\}$ and edge set $\{(C, s_1)(C, s_2) \mid s_1 s_2 \in E(S_{x|y})\}$. Let S' be the tree obtained from the disjoint union of $S_{y|x}$ with all the S_C over $C \in \mathcal{C}$ by adding the edges $(C, x)y$ for each $C \in \mathcal{C}$. See fig. 6.14. Let $f\colon V(S') \to V(S)$ be such that for every $z \in V(S')$,

$$f(z) = \begin{cases} s & \text{if } z = s \text{ for some } s \in V(S_{y|x}), \\ s & \text{if } z = (C, s) \text{ for some } s \in V(S_{x|y}) \text{ and } C \in \mathcal{C}. \end{cases}$$

Also, for every $z \in V(S')$, let

$$V'_z = \begin{cases} V_s & \text{if } z = s \text{ for some } s \in V(S_{y|x}), \\ V_s \cap V(C) & \text{if } z = (C, s) \text{ for some } s \in V(S_{x|y}) \text{ and } C \in \mathcal{C}. \end{cases}$$

Note that by construction, for every $z \in V(S')$, $V'_z \subseteq V_{f(z)}$. Moreover, $\mathcal{V}' = (S', (V'_z \mid z \in V(S')))$ is a tree decomposition of G.

To complete the construction, we specify a collection $\mathcal{Q}'$ of subtrees of S' whose vertex sets partition $V(S')$. Simultaneously, we define $g\colon \mathcal{Q}' \to \mathcal{Q}$. First, for every $Q \in \mathcal{Q}$ with $Q \cap V(S_{y|x}) \neq \emptyset$, let

$$Q' = S'\left[(V(Q) \cap V(S_{y|x})) \cup \bigcup_{C \in \mathcal{C}} \{(C, z) \mid z \in V(Q) \cap V(S_{x|y})\}\right] \text{ and } g(Q') = Q.$$

Next, for every $Q \in \mathcal{Q}$ with $Q \subseteq V(S_{x|y})$, for every $C \in \mathcal{C}$, let

$$Q_C = S'\left[\{(C, z) \mid z \in Q\}\right] \text{ and } g(Q_C) = Q.$$

Finally, let

$$\mathcal{Q}' = \left\{ Q' \mid Q \in \mathcal{Q} \text{ with } V(Q) \cap V(S_{y|x}) \neq \emptyset \right\} \cup \bigcup_{C \in \mathcal{C}} \left\{ Q_C \mid V(Q) \subseteq V(S_{x|y}) \right\}.$$

It follows that $\mathcal{Q}'$ is a collection of subtrees of S' whose sets of vertices partition $V(S')$ and and $f(V(Q')) = Q = g(Q)$ for every $Q' \in \mathcal{Q}'$. In other words, (r2) holds. Note that (r1) also holds, and thus, $(\mathcal{V}', \mathcal{Q}')$ refines $(\mathcal{V}, \mathcal{Q})$. Moreover, by transitivity of the refinement relation, $(\mathcal{V}', \mathcal{Q}')$ refines $(\mathcal{U}, \mathcal{P})$.

To conclude the proof, we show that $n(\mathcal{V}')$ is less than $n(\mathcal{V})$ in the lexicographic order, which leads to a contradiction. For every $s \in V(S_{x|y})$, we have $\sum_{C \in \mathcal{C}} |V'_{(C,s)}| = |V_s|$. Thus, either $|V'_{(C,s)}| < |V_s|$ for every $C \in \mathcal{C}$ or there is exactly one $C \in \mathcal{C}$ with $|V'_{(C,s)}| = |V_s|$ and $|V'_{(D,s)}| = 0$ for all $D \in \mathcal{C} \setminus \{C\}$. Moreover, for every $s \in V(S_{y|x})$, we have $V_s = V'_s$. Since G is connected, V_x intersects every connected component in $\mathcal{C}$. Also, note that we supposed $|\mathcal{C}| \geqslant 2$. Consider the maximum integer i_0 such that there exists $s \in V(S_{x|y})$ with $|V_s| = i_0$ and V_s intersects at least two connected components in $\mathcal{C}$. From the construction, we obtain that $n_i(\mathcal{V}') = n_i(\mathcal{V})$ for every integer i with $i > i_0$ and $n_{i_0}(\mathcal{V}') < n_{i_0}(\mathcal{V})$. We conclude that $n(\mathcal{V}')$ is less than $n(\mathcal{V})$ in the lexicographic order. $\square$

6.9.2 Centered colorings in K_t-minor-free graphs

In this section, we introduce the main ideas behind the proofs of our bounds on centered chromatic numbers, and in particular the notion of (q, c)-good coloring, and we apply them to show that K_t-minor-free graphs have q-centered chromatic numbers in $\mathcal{O}(q^{t-1})$ (Theorem 1.32).

6.9.2.1 The general idea

First, let us discuss the centered colorings of graphs of bounded treewidth. Let w and t be positive integers with $t \geqslant 2$, and let G be a graph with $\text{tw}(G) \leqslant w$. Pilipczuk and Siebertz [PS21], showed that $\text{cen}_p(G) \leqslant \binom{q+w}{w} = \mathcal{O}(q^w)$. They used elimination orderings of tree decompositions to localize potential centers in a structured way. To state this stronger version of the result, let us define a variant of centered colorings in ordered graphs. Let σ be an ordering of $V(G)$. A coloring φ of G is a *q-centered coloring* of (G, σ) if for every connected subgraph H of G, either $|\varphi(H)| > q$ or $\min_\sigma V(H)$ is a φ-center of $V(H)$. We denote by $\text{cen}_q(G, \sigma)$ the least nonnegative integer k such that (G, σ) admits a q-centered coloring using k colors.

Theorem 6.53 ([PS21])**.** *Let w be a positive integer, let G be a graph, and let $\mathcal{W}$ be a tree decomposition of G of width at most w. For every elimination ordering σ of $\mathcal{W}$ and for every positive integer q,*

$$\text{cen}_q(G, \sigma) \leqslant \binom{q+w}{w}.$$

The bound $\text{cen}_q(G) \leqslant \binom{q+w}{w}$ for graphs with $\text{tw}(G) \leqslant w$ can be used to obtain a good upper-bound of $\mathcal{O}(q^{t-2})$ on the q-centered chromatic number of K_t-minor-free graphs of bounded treewidth. To this end, we again use Theorem 6.28. Let G be a K_t-minor-free with $\text{tw}(G) \leqslant w$ and let $\mathcal{P}$ be a partition of $V(G)$ given by Theorem 6.28, that is, $G/\mathcal{P}$ has treewidth at most $t-2$ and each part of $\mathcal{P}$ has at most $w+1$ elements. Let ξ be a coloring of $G/\mathcal{P}$ given by theorem 6.53, and let ρ be a coloring of G using at most $w+1$ colors that is injective on each $P \in \mathcal{P}$. For every $P \in \mathcal{P}$ and $u \in P$, let $\zeta(u) = (\xi(P), \rho(u))$. To show that ζ is indeed a q-centered coloring of G, let H be a connected subgraph of G such that $|\zeta(V(H))| \leqslant q$. The parts of $\mathcal{P}$ that intersect $V(H)$ induce a connected subgraph of $G/\mathcal{P}$, thus, there is a ξ-center $P \in \mathcal{P}$ of the set of parts of $\mathcal{P}$ intersecting $V(H)$. Now, since ρ is injective on P, any vertex in P is a ζ-center of $V(H)$.

The previous paragraph sketches the proof of the statement like in Theorem 1.32 but for K_t-minor-free graphs of bounded treewidth. The essential property of the parts in $\mathcal{P}$ we used is that their sizes are bounded. There is no hope of obtaining such a partition $\mathcal{P}$, with each part of size bounded by a constant, in the general case of K_t-minor-free graphs. This forces us to relax the condition on parts. Instead of bounding the maximal size of a part, we want to maintain enough structural information. We accomplish this goal with an extra pre-coloring ρ of vertices of G to mimic the proof as in the paragraph above. Since ρ uses $\mathcal{O}(q)$ colors, this ultimately gives the bound $\mathcal{O}(q^{t-1})$ instead of $\mathcal{O}(q^{t-2})$. Below is a precise technical statement that we prove later in the chapter and a simple argument that the statement together with Theorem 6.53 implies Theorem 1.32.

Lemma 6.54. *For every positive integer t, there exists a positive integer $c_{6.54}(t)$ such that for every K_t-minor-free graph G and every positive integer q, there exists a partition $\mathcal{P}$ of $V(G)$, a tree decomposition $\mathcal{W}$ of $G/\mathcal{P}$ of width at most $t-2$, an elimination ordering $\sigma = (P_1, \dots, P_\ell)$ of $\mathcal{W}$, and a coloring ρ of G using at most $c_{6.54}(t) \cdot (q+1)$ colors such that for every connected subgraph H of $G[\bigcup\{Q \in \mathcal{P} \mid Q \geqslant_\sigma P\}]$ with $V(H) \cap P \neq \emptyset$, either $|\rho(V(H))| > q$, or $V(H) \cap P$ has a ρ-center.*

Proof of Theorem 1.32. Let $c_{6.54}(t)$ be the constant from Lemma 6.54 and let $c = 2^{t-1} \cdot c_{6.54}(t)$. Let G be a K_t-minor-free graph and let q be a positive integer. Let $\mathcal{P}$, $\mathcal{W}$, σ, and ρ be obtained by Lemma 6.54. Let ξ be a q-centered coloring of $G/\mathcal{P}$ using at most $\binom{p+t-2}{t-2}$ colors obtained from Theorem 6.53 applied to $G/\mathcal{P}$, $\mathcal{W}$, σ, and q. We define a coloring ζ of G in the following way. For every $P \in \mathcal{P}$ and $u \in P$,

$$\zeta(u) = (\xi(P), \rho(u)).$$

Note that

$$|\zeta(V(G))| \leqslant \binom{q+t-2}{t-2} \cdot c_{6.54}(t) \cdot (q+1) \leqslant c_{6.54}(t) \cdot (q+1)^{t-1} \leqslant c \cdot q^{t-1}.$$

It remains to show that ζ is a q-centered coloring of G. Let H be a connected subgraph of G with $|\zeta(V(H))| \leqslant q$. Let $\mathcal{P}_H = \{P \in \mathcal{P} \mid P \cap V(H) \neq \emptyset\}$ and let $P_H = \min_\sigma \mathcal{P}_H$. See fig. 6.15. Since $|\xi(\mathcal{P}_H)| \leqslant |\zeta(V(H))| \leqslant q$ and ξ is a q-centered coloring of $(G/\mathcal{P}, \sigma)$, P_H is a ξ-center of $\mathcal{P}_H$. So $\zeta(u) \neq \zeta(v)$ for all $u \in V(H) \cap P_H$ and $v \in V(H) \setminus P_H$. Now, H is a connected subgraph of $G[\bigcup\{Q \in \mathcal{P} \mid Q \geqslant_\sigma P_H\}]$ with $V(H) \cap P_H \neq \emptyset$ by definition of P_H. Since $|\rho(V(H))| \leqslant |\zeta(V(H))| \leqslant q$, there exists a ρ-center u in $V(H) \cap P_H$. It follows that u is a ζ-center in $V(H)$. We conclude that ζ is indeed a q-centered coloring of G. □

The key property of a graph G required for the idea in [ISW24] to work is that for every family $\mathcal{F}$ of connected subgraphs of G, there is a hitting set of $\mathcal{F}$ of size bounded by a function of the packing number[6] of $\mathcal{F}$. The parts in the final asserted partition are obtained as hitting sets of certain families. Graphs of bounded treewidth satisfy this property as witnessed by Lemma 1.17.

[6] The *packing number* of a family $\mathcal{F}$ of connected subgraphs of a graph G is the maximum number of pairwise vertex-disjoint members of $\mathcal{F}$.

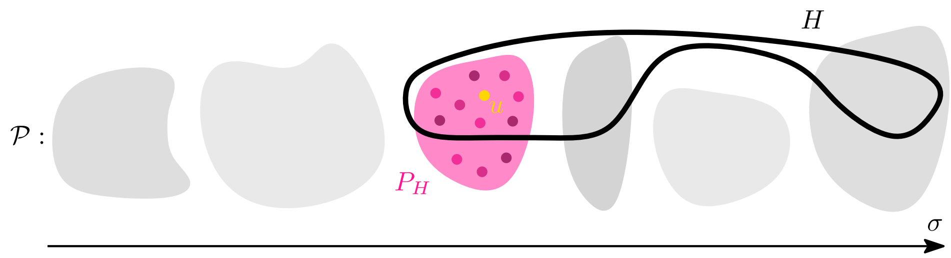


Figure 6.15: Finding a ζ-center of $V(H)$ in P_H.

However, K_t-minor-free graphs do not admit such a Helly-type property[7]. The observation that will eventually lead to the proof of Theorem 1.32 is that we do not need these hitting sets to have bounded size, as long as we can find a suitable coloring for them.

Let c be a positive integer. A coloring φ of G is (q,c)-*good* if for every subgraph G_0 of G, for every family $\mathcal{F}$ of connected subgraphs of G_0, for every positive integer d, if there are no $d+1$ pairwise disjoint members of $\mathcal{F}$, then there exists $Z \subseteq V(G_0)$ such that

(qc1) $V(F) \cap Z \neq \emptyset$ for every $F \in \mathcal{F}$;

(qc2) for every connected component C of $G_0 - Z$, $N_{G_0}(V(C))$ intersects at most two connected components of $G_0 - V(C)$;

(qc3) for every $P \subseteq Z$, there exists $R \subseteq Z$ with

- (a) $P \subseteq R$,
- (b) $\mathrm{cen}_p(G_0, \varphi, R) \leqslant c \cdot d$, and
- (c) $|R \setminus P| \leqslant c \cdot d$.

Note that (qc3) applied for $P = Z$ yields the following

(qc4) $\mathrm{cen}_q(G_0, \varphi, Z) \leqslant c \cdot d$.

Moreover, for every graph G, for every graph H, if φ is a (q,c)-good of G, then $\varphi|_{V(H)}$ is a (q,c)-good coloring of H. A crucial example of (q,c)-good coloring is given by the following lemma, which will cover the bounded treewidth case.

Lemma 6.55. *Let q be a positive integer and let G be a graph. There is a $(q,2)$-good coloring of G using $\mathrm{tw}(G)+1$ colors.*

Proof. Let $(T, (W_x \mid x \in V(T)))$ be a natural tree decomposition of G of width $\mathrm{tw}(G)$. Such a tree decomposition exists by Lemma 5.3. Let $\varphi\colon V(G) \to [\mathrm{tw}(G)+1]$ be such that for every $x \in V(T)$, $\varphi|_{W_x}$ is injective.

We now show that φ is a (q,c)-good of G. Let G' be a subgraph of G, let d be a positive integer, and let $\mathcal{F}$ be a family of connected subgraphs of G' with no $d+1$ pairwise disjoint members. By Lemma 1.17 and Lemma 5.4, there is a set $X \subseteq V(T)$ of size at most $2d-1$ such that for $Z = \bigcup_{x \in X} V(T)$,

[7]Consider the $n \times n$ planar grid G and the family $\mathcal{F}$ consisting of all the unions of one row and one column in the grid. The packing number of $\mathcal{F}$ is 1 but there is no hitting set of $\mathcal{F}$ with less than n elements. Additionally, each planar grid is K_5-minor-free.

(i) $V(F) \cap Z \neq \emptyset$ for every $F \in \mathcal{F}$, and so (qc1) holds; and

(ii) for every connected component C of $G' - Z$, $N_{G'}(V(C))$ intersects at most two connected components of $G' - V(C)$, and so (qc2) holds.

Now, let $\psi\colon Z \to X$ such that $z \in W_{\psi(x)}$ for every $z \in Z$. Note that $\varphi|_Z \times \psi$ is injective. Hence, for every connected subgraph H of G intersecting Z, any vertex in $V(H) \cap Z$ is a $(\varphi|_Z \times \psi)$-center of $V(H) \cap Z$.

Finally, for every $P \subseteq Z$, $\mathrm{cen}_q(G_0, \varphi, P) \leqslant |P| \leqslant 2 \cdot d$. This shows (qc3), and proves that φ is a $(q, 2)$-good coloring of G. □

One of the crucial steps to bound centered chromatic numbers in minor-closed classes of graphs is the following lemma proved in Section 6.9.3.

Lemma 6.56. *For every positive integer t, there exists a positive integer $c_{6.56}(t)$ such that for every K_t-minor-free graph G and every positive integer q, G admits a $(q, c_{6.56}(t))$-good coloring using $q + 1$ colors.*

6.9.2.2 From good colorings to centered colorings

In this section, we prove Lemma 6.54 assuming Lemma 6.56 (proved in section 6.9.3). The lemma below is an inductive setup of the proof and is inspired by the framework given in [ISW24].

Lemma 6.57. *Let t be an integer with $t \geqslant 2$ and let q, c be positive integers. Let G be a K_t-minor-free graph, let r be an integer with $0 \leqslant r \leqslant t - 2$, and let $R_1, \dots, R_r$ be pairwise disjoint subsets of $V(G)$ with $|R_i| \in \{1, 2\}$ for every $i \in [r]$. Let φ be a (q, c)-good coloring of $G - \bigcup_{i \in [r]} R_i$. There are a partition $\mathcal{P}$ of $V(G)$, a tree decomposition $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ of $G/\mathcal{P}$ of width at most $t - 2$, and an elimination ordering $\sigma = (P_1, \dots, P_\ell)$ of $\mathcal{W}$ such that*

(a) *$P_i = R_i$ for every $i \in [r]$;*

(b) *there exists $s \in V(T)$ such that $R_1, \dots, R_r \in W_s$;*

(c) *for every $P \in \mathcal{P} \setminus \{R_1, \dots, R_r\}$, there is a coloring $\psi_P\colon P \to [c \cdot 2^{t-2}(t-1)]$ such that for every connected subgraph H of $G\left[\bigcup\{Q \in \mathcal{P} \mid Q \geqslant_\sigma P\}\right]$ with $V(H) \cap P \neq \emptyset$, either $|\varphi(V(H))| > q$, or $V(H) \cap P$ has a $(\varphi \times \psi_P)$-center.*

Proof. Let $\mathcal{R} = \{R_1, \dots, R_r\}$, and let $R = \bigcup_{i \in [r]} R_i$. We proceed by induction on $(|V(G - R)|, r)$ in the lexicographic order. For the base case, assume that $V(G - R) = \emptyset$. Then we set $\mathcal{P} = \mathcal{R}$ and $\sigma = (R_1, \dots, R_r)$. Let $T = K_1$, and let $W_s = \mathcal{P}$ for s the unique vertex of T. Then (a) and (b) hold by construction, and (c) is vacuously true. From now on, we assume $V(G - R) \neq \emptyset$.

If $r < t - 2$, then we set R_{r+1} to be an arbitrary singleton in $V(G - R)$ and we apply induction on $G, (R_1, \dots, R_{r+1}), \varphi|_{V(G - R - R_{r+1})}$. This gives a partition $\mathcal{P}$ of $V(G)$, a tree decomposition $\mathcal{W}$ of $G/\mathcal{P}$, and an elimination ordering σ of $\mathcal{W}$ satisfying all three items. Items (a) and (b) are now clear by induction. Item (c) is clear for all $P \in \mathcal{P} \setminus \{R_1, \dots, R_r, R_{r+1}\}$, thus, it suffices to argue it for $P = R_{r+1}$. Recall that $R_{r+1} = \{v\}$ for some $v \in V(G)$. We set $\varphi_{R_{r+1}}(v) = 1$. Observe that in this case for every subgraph H of G intersecting R_{r+1}, $V(H) \cap R_{r+1}$ has a $(\varphi \times \psi_{R_{r+1}})$-center. This completes the proof in the case $r < t - 2$. Therefore, from now on, we assume $r = t - 2$.

Next, suppose that $G - R$ is not connected, and let $\mathcal{C}$ be the family of all the connected components of $G - R$. Fix some $C \in \mathcal{C}$ and let $G_C = G[V(C) \cup R]$. By the induction hypothesis applied to G_C, $(R_1, \dots, R_{t-2})$, and $\varphi|_{V(C)}$, there are a partition $\mathcal{P}_C$ of $V(G_C)$, a tree decomposition $\mathcal{W}_C = (T_C, (W_{C,x} \mid x \in V(T_C)))$ of $G_C/\mathcal{P}_C$ of width at most $t-2$ and an elimination ordering $\sigma_C = (P_{C,1}, \dots, P_{C,\ell_C})$ of $\mathcal{W}_C$ such that

(a') $P_{C,i} = R_i$ for every $i \in [r]$;

(b') there exists $s_C \in V(T_C)$ such that $R_1, \dots, R_{t-2} \in W_{C,s_C}$;

(c') for every $P \in \mathcal{P}_C \setminus \{R_1, \dots, R_{t-2}\}$, there is a coloring $\psi_{C,P} \colon P \to [c \cdot 2^{t-2}(t-1)]$ such that for every connected subgraph H of $G_C\left[\bigcup\{Q \in \mathcal{P}_C \mid Q \geqslant_{\sigma_C} P\}\right]$ with $V(H) \cap P \neq \emptyset$, either $|\varphi(V(H))| > q$, or $V(H) \cap P$ has a $(\varphi \times \psi_{C,P})$-center.

Then, let

$$\mathcal{P} = \bigcup_{C \in \mathcal{C}} \mathcal{P}_C,$$

and let σ be the concatenation of

$$(R_1, \dots, R_{t-2}), (P_{C_1,t-1}, \dots, P_{C_1,\ell_C}), \dots, (P_{C_a,t-1}, \dots, P_{C_a,\ell_C}),$$

for an arbitrary ordering $(C_1, \dots, C_a)$ of $\mathcal{C}$. Let T be obtained from the disjoint union of T_C over all $C \in \mathcal{C}$ by adding a new vertex s with the neighborhood $\{s_C \mid C \in \mathcal{C}\}$. For every $C \in \mathcal{C}$ and every $x \in V(T_C)$, let $W_x = W_{C,x}$. Additionally, let $W_s = \{R_1, \dots, R_{t-2}\}$. By construction, $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ is a tree decomposition of $G/\mathcal{P}$ of width at most $t-2$ and σ is an elimination ordering of $\mathcal{W}$. Moreover, (a) and (b) are clearly satisfied. It remains to show that (c) holds. Let $P \in \mathcal{P} \setminus \{R_1, \dots, R_{t-2}\}$. There exists $C \in \mathcal{C}$ such that $P \in \mathcal{P}_C$. Let $\psi_P = \psi_{C,P}$. Let H be a connected subgraph of $G\left[\bigcup\{Q \in \mathcal{P} \mid Q \geqslant_\sigma P\}\right]$ with $V(H) \cap P \neq \emptyset$. Note that H is a connected subgraph of $G_C\left[\bigcup\{Q \in \mathcal{P}_C \mid Q \geqslant_{\sigma_C} P\}\right]$. Hence, by (c') either $|\varphi(V(H))| > q$ or $V(H) \cap P$ has a $(\varphi \times \psi_{C,P})$-center u. In the former case, we are immediately satisfied, and in the latter case, u is also a $(\varphi \times \psi_P)$-center of $V(H) \cap P$. Therefore, (c) holds, which concludes the case where $G - R$ is not connected. From now on, we assume that $G - R$ is connected.

Suppose that there exists $i \in [t-2]$ such that $N_G(R_i) \cap V(G - R) = \emptyset$. In this case, we call induction on $G - R_i, (R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_{t-2}), \varphi$. We deduce that there is a partition $\mathcal{P}'$ of $V(G - R_i)$, a tree decomposition $\mathcal{W}' = (T', (W_x \mid x \in V(T)))$ of $(G - R_i)/\mathcal{P}'$ of width at most $t-2$, and an elimination ordering $\sigma' = (P'_1, \dots, P'_{\ell'})$ of $\mathcal{W}'$ satisfying all three items. In particular there exists $s' \in V(T')$ such that $R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_{t-2} \in W_{s'}$. Let $\mathcal{P} = \mathcal{P}' \cup \{R_i\}$ and let $\sigma = (P'_1, \dots, P'_{i-1}, R_i, P'_{i+1}, \dots, P'_{\ell'})$. Let T be obtained from T' by adding a new neighbor s of s'. Finally, let $W_s = \{R_1, \dots, R_{t-2}\}$ and let $W_x = W'_x$ for every $x \in V(T')$. We obtain that $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ is a tree decomposition of $G/\mathcal{P}$ of width at most $t-2$ and σ is an elimination ordering of $\mathcal{W}$. It is immediate from the induction hypothesis that $\mathcal{P}$, $\mathcal{W}$, and σ satisfy (a)-(c). From now on, we assume that for every $i \in [t-2]$, there is an edge between $V(G - R)$ and R_i.

This completes the series of simple reductions. As a result we can assume the following: $V(G - R) \neq \emptyset$, $r = t-2$, $G - R$ is connected, and for every $i \in [t-2]$, there is an edge between $V(G - R)$ and R_i. We proceed with the main argument.

Let $\mathcal{F}$ be the family of all connected subgraphs H of $G - R$ such that $N_G(R_i) \cap V(H) \neq \emptyset$ for every $i \in [t-2]$. We claim that there are no $2^{t-2}(t-1)+1$ pairwise disjoint members of

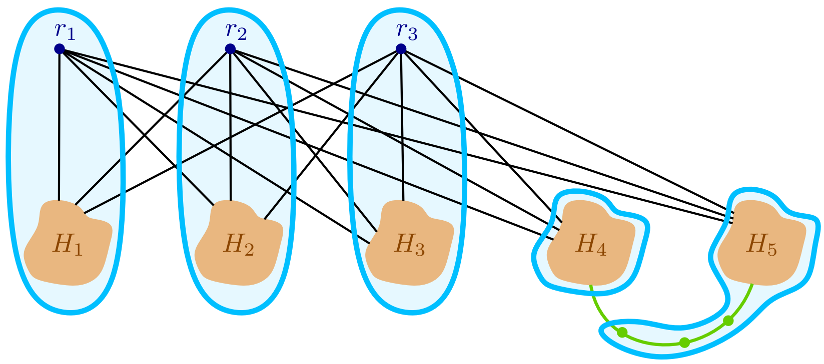


Figure 6.16: After pigeonholing pairwise disjoint members of $\mathcal{F}$, we obtain a situation as in the figure. Here, $t = 5$. The model of K_t is constructed in blue.

$\mathcal{F}$. Suppose to the contrary that there are $2^{t-2}(t-1)+1$ pairwise disjoint members of $\mathcal{F}$. Since for each $i \in [t-2]$, $|R_i| \leqslant 2$, and every member of $\mathcal{F}$ has a neighbor in R_i, by the pigeonhole principle, there exist pairwise disjoint $H_1, \dots, H_t \in \mathcal{F}$ and there exists $r_i \in R_i$ for each $i \in [t-2]$ such that for all $j \in [t]$ and $i \in [t-2]$, we have that H_j contains a neighbor of r_i in G. Since $G - R$ is connected, there exist distinct $i_1, i_2 \in [t]$ such that there is a $(V(H_{i_1}), V(H_{i_2}))$-path in $G - R$ internally disjoint from $\bigcup_{j \in [t]} V(H_j)$. Let A be the set of internal vertices of this path. Without loss of generality, suppose that $i_1 = t-1$ and $i_2 = t$. Then

$$(V(H_1) \cup \{r_1\}, \dots, V(H_{t-2}) \cup \{r_{t-2}\}, V(H_{t-1}), V(H_t) \cup A)$$

is a model of K_t in G (see fig. 6.16), which contradicts the assumption as G is K_t-minor-free. This contradiction concludes the proof that $\mathcal{F}$ has no $2^{t-2}(t-1)+1$ pairwise disjoint members.

Since φ is a (q,c)-good coloring of $G - R$, it follows that there exists a set $Z \subseteq V(G-R)$ and $\psi_Z\colon Z \to [c \cdot 2^{t-2}(t-1)]$ such that

(qc1') $V(F) \cap Z \neq \emptyset$ for every $F \in \mathcal{F}$;

(qc2') for every connected subgraph H of $G - R$ with $V(H) \cap Z \neq \emptyset$, either $|\varphi(V(H))| > q$, or $V(H) \cap Z$ has a $(\varphi \times \psi_Z)$-center;

(qc3') for every connected component C of $G - R - Z$, $N_{G-R}(V(C))$ intersects at most two connected components of $G - R - V(C)$.

Note that $G - R \in \mathcal{F}$, and so $\mathcal{F}$ is nonempty. In particular, Z is nonempty. Let $\mathcal{C}$ be the family of all the connected components of $G - R - Z$.

Fix some $C \in \mathcal{C}$. Since $V(C) \cap Z = \emptyset$, by (qc1'), we have, $C \notin \mathcal{F}$. In particular, there exists $i \in [t-2]$ such that $N_G(R_i) \cap V(C) = \emptyset$. Let G_C be obtained from $G - R_i$ by contracting each connected component of $G - R - V(C)$ into a single vertex. See fig. 6.17. Let Z_C be the set of the resulting contracted vertices. Since $Z \neq \emptyset$, we have $1 \leqslant |Z_C|$, and since $G - R$ is connected, by (qc3'), $|Z_C| \leqslant 2$. Moreover, G_C is a minor of G, thus, G_C is K_t-minor-free. Let $\mathcal{R}_C = (\mathcal{R} \setminus \{R_i\}) \cup \{Z_C\}$ and let $(R_{C,1}, \dots, R_{C,t-2})$ be an arbitrary ordering of $\mathcal{R}_C$. Since $Z \neq \emptyset$, we have

$$|V(G_C - \bigcup \mathcal{R}_C)| < |V(G-R)|.$$

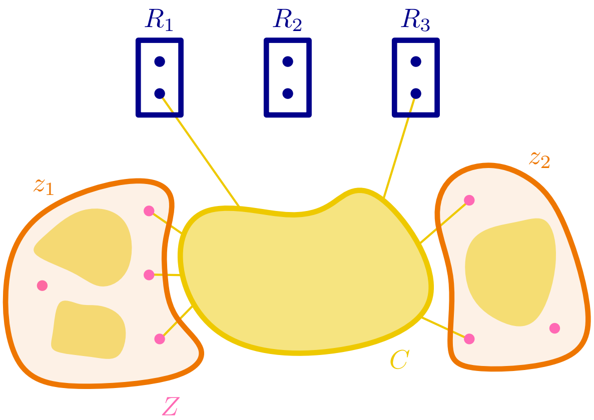


Figure 6.17: The pink vertices depict the set Z. The yellow pieces are connected components of $G - R - Z$. In the figure, $i = 2$, that is, C does not have a neighbor in R_2. The set Z_C consists of two vertices z_1 and z_2 obtained by contracting the orange parts. In order to apply induction, we replace R_2 with $\{z_1, z_2\}$.

Hence, by the induction hypothesis applied to G_C, $(R_{C,1}, \ldots, R_{C,t-2})$, and $\varphi|_{V(C)}$, there is a partition $\mathcal{P}_C$ of $V(G_C)$, a tree decomposition $\mathcal{W}_C = (T_C, (W_{C,x} \mid x \in V(T_C)))$ of $G_C/\mathcal{P}_C$ of width at most $t-2$, and an elimination ordering $\sigma_C = (P_{C,1}, \ldots, P_{C,\ell_C})$ of $\mathcal{W}_C$ such that

(a”) $P_{C,i} = R_{C,i}$ for every $i \in [t-2]$;

(b”) there exists $s_C \in V(T_C)$ such that $R_{C,1}, \ldots, R_{C,r} \in W_{C,s_C}$; and

(c”) for every $P \in \mathcal{P}_C \setminus \{R_{C,1}, \ldots, R_{C,t-2}\}$, there is a coloring $\psi_{C,P} \colon P \to [c \cdot 2^{t-2}(t-1)]$ such that for every connected subgraph H of $G_C\left[\bigcup\{Q \in \mathcal{P}_C \mid Q \geqslant_{\sigma_C} P\}\right]$ with $V(H) \cap P \neq \emptyset$, either $|\varphi(V(H))| > q$, or $V(H) \cap P$ has a $(\varphi \times \psi_{C,P})$-center.

Then let

$$\mathcal{P} = \bigcup_{C \in \mathcal{C}} (\mathcal{P}_C \setminus \mathcal{R}_C) \cup \mathcal{R} \cup \{Z\},$$

and let σ be the concatenation of

$$(R_1, \ldots, R_{t-2}, Z), (P_{C_1,t-1}, \ldots, P_{C_1,\ell_{C_1}}), \ldots, (P_{C_a,t-1}, \ldots, P_{C_a,\ell_{C_a}}),$$

for an arbitrary ordering $(C_1, \ldots, C_a)$ of $\mathcal{C}$. Let T be obtained from the disjoint union of T_C over all $C \in \mathcal{C}$ by adding a new vertex s with the neighborhood $\{s_C \mid C \in \mathcal{C}\}$. For every $C \in \mathcal{C}$ and every $x \in V(T_C)$, let $W_x = W_{C,x}$. Additionally, let $W_s = \{R_1, \ldots, R_{t-2}, Z\}$. By construction, $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ is a tree decomposition of $G/\mathcal{P}$ of width at most $t-2$ and σ is an elimination ordering of $\mathcal{W}$. Moreover, (a) and (b) are clearly satisfied.

It remains to show that (c) holds. Let $P \in \mathcal{P} \setminus \{R_1, \ldots, R_{t-2}\}$. First, suppose that $P \neq Z$. Then there exists $C \in \mathcal{C}$ such that $P \in \mathcal{P}_C$. We set $\psi_P = \psi_{C,P}$. Let H be a connected subgraph of $G\left[\bigcup\{Q \in \mathcal{P}, Q \geqslant_\sigma P\}\right]$ with $V(H) \cap P \neq \emptyset$. Then H is a connected subgraph of $G_C\left[\bigcup\{Q \in \mathcal{P}_C \mid Q \geqslant_{\sigma_C} P\}\right]$. Hence, by (c”) either $|\varphi(V(H))| > q$ or $V(H) \cap P$ has a $(\varphi \times \psi_{C,P})$-center u. In the former case, we are immediately satisfied, and in the latter case, u is also a $(\varphi \times \psi_P)$-center of $V(H) \cap P$. This yields (c) when $P \neq Z$. Finally, suppose that $P = Z$.

Let H be a connected subgraph of $G\left[\bigcup\{Q \in \mathcal{P} \mid Q \geqslant_\sigma Z\}\right] = G - R$. Now (c) follows directly from (qc2'). Summarizing, we constructed $\mathcal{P}$, $\mathcal{R}$, and σ satisfying (a)-(c), as desired. □

Proof of Lemma 6.54. Let t be a positive integer and set $c_{6.54}(t) = c_{6.56}(t) \cdot 2^{t-2}(t-1)$. Let G be a K_t-minor-free graph and let q be a positive integer. By Lemma 6.56, there exists a $(q, c_{6.56}(t))$-good coloring φ of G using at most $q+1$ colors. Next, we apply Lemma 6.57 with $c = c_{6.56}(t)$ and $r = 0$. We obtain a partition $\mathcal{P}$ of $V(G)$, a tree decomposition $\mathcal{W}$ of $G/\mathcal{P}$ of width at most $t-2$, and an elimination ordering $\sigma = (P_1, \dots, P_\ell)$ of $\mathcal{W}$ such that for every $P \in \mathcal{P}$, there is a coloring $\psi_P\colon P \to [c_{6.56}(t) \cdot 2^{t-2}(t-1)]$ such that for every connected subgraph H of $G\left[\bigcup\{Q \in \mathcal{P} \mid Q \geqslant_\sigma P\}\right]$ with $V(H) \cap P \neq \emptyset$, either $|\varphi(V(H))| > q$, or $V(H) \cap P$ has a $(\varphi \times \psi_P)$-center. For every $P \in \mathcal{P}$ and $u \in P$, let $\rho(u) = (\varphi(u), \psi_P(u))$. Thus, ρ is a coloring of G using at most $c_{6.54}(t) \cdot (q+1)$ colors and satisfying the assertion of Lemma 6.54. □

6.9.3 Constructing good colorings

In this subsection, we prove Lemma 6.56, namely, we show that for every positive integer t, there exists a positive integer $c_{6.56}(t)$ such that for every K_t-minor-free graph G and every positive integer q, G admits a $(q, c_{6.56}(t))$-good coloring using $q+1$ colors.

For purposes of a short discussion, consider a simpler variant of good colorings, where we drop (qc2), which is required in the real definition for very technical reasons. Observe that in the definition of good colorings, a subgraph G_0 occurs only in (qc2), hence, if we ignore this item, we can also assume that $G_0 = G$. We say that a coloring is a *simple* (q, c)-good coloring if it admits the relaxed definition above.

For graphs of bounded treewidth, there is a straightforward way of constructing simple good colorings. Let G be a graph. We can set φ to use only one color, and for a family $\mathcal{F}$ of connected subgraphs of G having no $d+1$ pairwise disjoint members we take Z as given by Lemma 1.17, so (qc1) is satisfied. For an injective $\psi\colon Z \to [(\mathrm{tw}(G)+1) \cdot d]$, (qc4) is clearly satisfied. This gives a $(q, \mathrm{tw}(G)+1)$-good coloring of G using one color for every positive integer q. A true inspiration for the definition of good colorings comes from the case of graphs of bounded layered treewidth.

Let G be a graph, let q be a positive integer, and let $\mathrm{ltw}(G)$ be witnessed by $(\mathcal{W}, \mathcal{L})$, where $\mathcal{W} = (T, (W_x \mid x \in V(T)))$ and $\mathcal{L} = (L_i \mid i \in \mathbb{N})$. For all $i \in \mathbb{N}$ and $v \in L_i$, we define $\varphi(v) = i \bmod (q+1)$. We claim that φ is a simple $(p, \mathrm{ltw}(G))$-good coloring of G. Let $\mathcal{F}$ be a family of connected subgraphs of G with no $d+1$ pairwise disjoint members. Let Z be the union of at most d bags of $\mathcal{W}$ that is a hitting set of $\mathcal{F}$ in G (given by Lemma 1.17). In particular, $|Z \cap L_i| \leqslant \mathrm{ltw}(G) \cdot d$ for every $i \in \mathbb{N}$. Let $\psi\colon Z \to [\mathrm{ltw}(G) \cdot d]$ be any injective coloring on $Z \cap L_i$ for every $i \in \mathbb{N}$. It remains to verify (qc4). Let H be a connected subgraph of G with $V(H) \cap Z \neq \emptyset$. Since H is connected and $\mathcal{L}$ is a layering of G, $V(H)$ intersects a set of consecutive layers in $\mathcal{L}$. If this set has at least $q+1$ elements, then $|\varphi(V(H))| > q$, and otherwise any element of $V(H) \cap Z$ is a $(\varphi \times \psi)$-center of $V(H) \cap Z$.

To lift this idea to the class of general K_t-minor-free graphs, we again use a layered RS-decomposition (see Section 6.1).

Given a layered RS-decomposition of a graph G of bounded width, the strategy to find a simple good coloring of G is the following. Let q be a positive integer. The intention is to mimic the proof for the bounded layered treewidth case. That is, ignoring the overlapping between the bags of $\mathcal{W}$, we set $\varphi(v) \equiv i \bmod (q+1)$ for every $i \in \mathbb{N}$ and every $v \in L_{x,i}$. We claim that φ is a simple

(q,c)-good coloring, where c depends only on the width of the given decomposition. Let $\mathcal{F}$ be a family of connected subgraphs of G with no $d+1$ pairwise disjoint members. We glue all tree decompositions in $\mathcal{D}$ in a natural way to obtain a tree decomposition $\mathcal{U}$ of G. The vertices of A_x do not occur in bags of tree decompositions of $\mathcal{D}$, but we can add them to all bags of $\mathcal{U}$ corresponding to W_x. The important property of $\mathcal{U}$ is that (lrs5) is preserved. Next, we apply Lemma 1.17 to $\mathcal{F}$ and $\mathcal{U}$ obtaining a hitting set Z of $\mathcal{F}$ in G, which is the union of at most d bags of $\mathcal{U}$. Note that there are also at most d bags of $\mathcal{W}$, whose union contains Z. Say that these bags correspond to the vertices in $X \subseteq V(T)$. The next step is to "disconnect" elements of Z in different bags of $\mathcal{W}$. To this end, we root T arbitrarily and we define B as the union of $\bigcup_{x\in X} A_x$ and the union of adhesions between W_x and the bag corresponding to the parent of x in $\mathcal{W}$ for all $x \in X$. In particular, the size of B is bounded. Finally, we define a coloring ψ of Z so that ψ is injective on B and ψ is injective on $L_{x,i} \cap Z$ but the colors used are disjoint from $\psi(B)$. Checking that ψ witnesses φ being a simple (q,c)-good coloring of G is very similar to the bounded layered treewidth case.

Proof of Lemma 6.56. Let t be a positive integer. Let $c_{\mathrm{LRS}}(t)$ be the constant from Theorem 6.1. We set

$$c_{6.56}(t) = c_{\mathrm{LRS}}(t)\,(12c_{\mathrm{LRS}}(t)+10)\,.$$

Let G be a K_t-minor-free graph, and q be a positive integer. By Theorem 6.1, G admits a layered RS-decomposition $(T,\mathcal{W},\mathcal{A},\mathcal{D},\mathcal{L})$ of width at most $c_{\mathrm{LRS}}(t)$. Let $\mathcal{W} = (W_x \mid x \in V(T))$, $\mathcal{A} = (A_x \mid x \in V(T))$, $\mathcal{D} = ((T_x, (D_{x,z} \mid z \in V(T_x))) \mid x \in V(T))$, and $\mathcal{L} = ((L_{x,i} \mid i \in \mathbb{N}) \mid x \in V(T))$. We root T in an arbitrary vertex $r \in V(T)$.

We define a coloring $\varphi\colon V(G) \to \{0,\dots,q\}$ as follows. For every $v \in V(G)$ let

$$\varphi(v) = \begin{cases} 0 & \text{if } v \in A_r,\\ i \bmod (q+1) & \text{if } v \in W_r \setminus A_r \text{ and } v \in L_{r,i},\\ 0 & \text{if } v \in A_x \setminus W_{\mathrm{p}(T,x)} \text{ for } x \in V(T)\setminus\{r\},\\ i \bmod (q+1) & \text{if } v \in W_x \setminus (A_x \cup W_{\mathrm{p}(T,x)}) \text{ and } v \in L_{x,i} \text{ for } x \in V(T)\setminus\{r\}. \end{cases}$$

It remains to show that φ is a $(q, c_{6.56}(t))$-good coloring of G. Fix a subgraph G_0 of G.

Now, we reduce to the case where G_0 is connected. Indeed, if G_0 is not connected, consider the family $\mathcal{C}$ of all the connected components of G_0. Let d be a nonnegative integer and let $\mathcal{F}$ be a family of connected subgraphs of G_0 such that there are no $d+1$ disjoint members of $\mathcal{F}$. For every $C \in \mathcal{C}$, let d_C be the smallest integer such that there are no d_C+1 disjoint members of $\mathcal{F}_C = \{F \in \mathcal{F} \mid F \subseteq C\}$. Clearly $d_C \leqslant d$. Assuming we can prove the result when G_0 is connected, we apply it for C and $\mathcal{F}_C$ and obtain a set $Z_C \subseteq V(C)$ and $\psi_C\colon Z_C \to [c\cdot d]$ such that (qc1)-(qc2) holds. Now take $Z = \bigcup_{C\in\mathcal{C}} Z_C$ and let ψ be defined by $\psi(u) = \psi_C(u)$ for every $C \in \mathcal{C}$ and every $u \in Z_C$. Recall that each $F \in \mathcal{F}$ is connected, so $F \subseteq C$ for some $C \in \mathcal{C}$, and therefore $Z_C \cap V(F) \neq \emptyset$. Thus, (qc1) holds. Since every connected subgraph H of G_0 is a subgraph of C for some $C \in \mathcal{C}$, (qc4) holds. Finally, (qc2) holds because for every connected component C' of $G - Z$, there exists $C \in \mathcal{C}$ such that $C' \subseteq C$.

Therefore, from now on we assume that G_0 is connected. We start by building a normal pair $(\mathcal{U},\mathcal{P})$ where $\mathcal{U}$ is a tree decomposition of G_0 obtained from "gluing" tree decompositions in $\mathcal{D}$ along the edges of T. This construction is depicted in Figure 6.18.

For all $x, y \in V(T)$ with $xy \in E(T)$, $W_x \cap W_y$ induces a clique in both $\mathrm{torso}_{G,\mathcal{W}}(W_x)$ and $\mathrm{torso}_{G,\mathcal{W}}(W_y)$, hence, there exist $z_{xy} \in V(T_x)$ and $z_{yx} \in V(T_y)$ such that $W_x \cap W_y =$

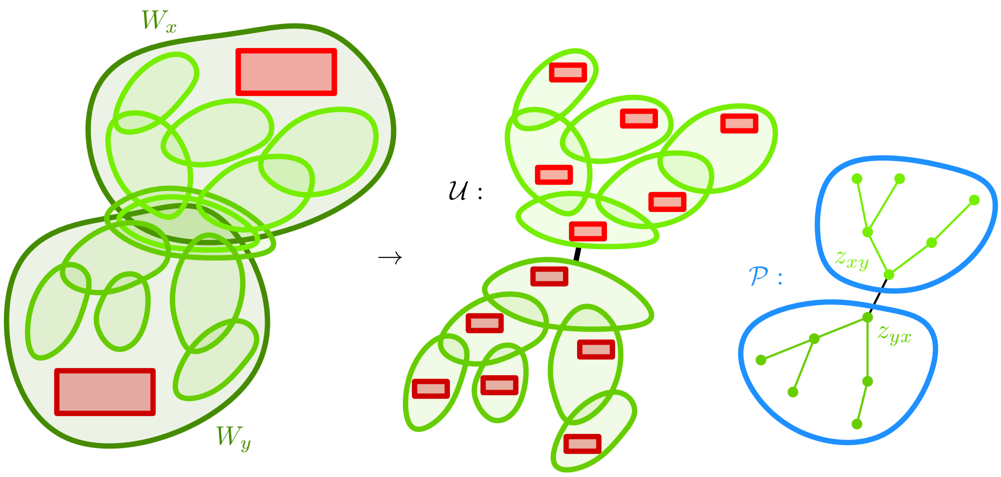


Figure 6.18: Tree decompositions $\mathcal{D}_x$ and $\mathcal{D}_y$ are "glued" in a natural way. Additionally, we add all corresponding apices (members of A_x and A_y) to each bag of the corresponding part in the tree decomposition $\mathcal{U}$. The partition $\mathcal{P}$ indicates from which of $\{T_x \mid x \in V(T)\}$ a given vertex comes from.

$D_{x,z_{xy}} \cap D_{y,z_{yx}}$. Let T_0 be the tree defined by

$$V(T_0) = \{(x,z) \mid x \in V(T), z \in V(T_x)\} \text{ and}$$
$$E(T_0) = \{(x,z_1)(x,z_2) \mid x \in V(T), z_1z_2 \in E(T_x)\} \cup \{(x,z_{xy})(y,z_{yx}) \mid xy \in E(T)\}.$$

In other words, T_0 is obtained from the disjoint union of T_x over $x \in V(T)$ by adding the edges between z_{xy} and z_{yx} for adjacent vertices x and y in T. Next, let $U_{(x,z)} = (D_{x,z} \cup A_x) \cap V(G_0)$ for every $(x,z) \in V(T_0)$. We claim that $\mathcal{U} = (T_0, (U_{(x,z)} \mid (x,z) \in V(T_0)))$ is a tree decomposition of G_0. For every $v \in V(G_0)$, $T_0[\{(x,z) \in V(T_0) \mid v \in U_{(x,z)}\}]$ is isomorphic to the connected subgraph of T_0 formed by replacing every vertex x of $T(v) = T[\{x \in V(T) \mid v \in W_x\}]$ with $T_x[\{z \in V(T_x) \mid v \in D_{x,z}\}]$ when $v \notin A_x$ or with T_x when $v \in A_x$, and every edge xy of $T(v)$ with $z_{xy}z_{yx}$. For every edge $uv \in E(G_0)$, there exists $x \in V(T)$ such that $u,v \in W_x$. If $u,v \in W_x \setminus A_x$, then there exists $z \in V(T_x)$ such that $u,v \in D_{x,z}$ and thus $u,v \in U_{(x,z)}$. If $u \in A_x$ and $v \in W_x \setminus A_x$, then for any $z \in V(T_x)$ with $v \in D_{x,z}$ we have $v \in U_{(x,z)}$ and $u \in A_x \cap V(G_0) \subseteq U_{(x,z)}$. If $u,v \in A_x$, then for every $z \in V(T_x)$, we have $u,v \in U_{(x,z)}$. Thus, indeed $\mathcal{U}$ is a tree decomposition of G_0.

Observe that $\mathcal{P} = \{T_0[\{x\} \times V(T_x)] \mid x \in V(T)\}$ is a collection of subtrees of T_0 whose vertex sets partition $V(T_0)$. Therefore, $(\mathcal{U}, \mathcal{P})$ is a normal pair of G_0.

By Lemma 6.52, there exists $\mathcal{V} = (S, (V_y \mid y \in V(S)))$ and $\mathcal{Q}$ such that $\mathcal{V}$ is natural and $(\mathcal{V}, \mathcal{Q})$ is a normal pair of G refining $(\mathcal{U}, \mathcal{P})$. Among all such pairs $(\mathcal{V}, \mathcal{Q})$, we take one with $|\mathcal{Q}|$ minimum. Let $f\colon V(S) \to V(T_0)$ and $g\colon \mathcal{Q} \to \mathcal{P}$ witness the refinement relation. For every $Q \in \mathcal{Q}$, let $x(Q) \in V(T)$ be such that $g(Q) = T_0[\{x(Q)\} \times V(T_{x(Q)})]$. See Figure 6.19.

In the next two claims, we show that the new tree decomposition $\mathcal{V}$ in some sense preserves small adhesions (only ones coming from $\mathcal{W}$) and small intersections with layers of the layerings in $\mathcal{L}$. In the first claim, we exploit the minimality of $(\mathcal{V}, \mathcal{Q})$.

Claim 6.56.1. *Let $y, y' \in V(S)$ and $Q, Q' \in \mathcal{Q}$ be such that $y \in V(Q)$, $y' \in V(Q')$ and $Q \neq Q'$. Then*
$$|V_y \cap V_{y'}| \leqslant c_{\mathrm{LRS}}(t).$$

Proof of the claim. By properties of tree decompositions, there is an edge zz' in S such that $V_y \cap V_{y'} \subseteq V_z \cap V_{z'}$. Therefore, without loss of generality, we assume that yy' is an edge in S. We argue that $g(Q) \neq g(Q')$. Suppose to the contrary that $g(Q) = g(Q')$. Since yy' is an edge in S, the subgraphs Q and Q' are adjacent in S, and thus, $Q'' = S[V(Q) \cup V(Q')]$ is a connected subgraph of S. Observe that $(\mathcal{V}, \mathcal{Q} \setminus \{Q, Q'\} \cup \{Q''\})$ is a normal pair which refines $(\mathcal{U}, \mathcal{P})$ as witnessed by f and g' defined by $g'(P) = g(P)$ for every $P \in \mathcal{Q} \setminus \{Q, Q'\}$ and $g'(Q'') = g(Q) = g(Q')$. However, this contradicts the minimality of $|\mathcal{Q}|$. We obtain that indeed $g(Q) \neq g(Q')$, and so, $x(Q) \neq x(Q')$. In particular,
$$|V_y \cap V_z| \leqslant |U_{(x(Q),f(y))} \cap U_{(x(Q'),f(y'))}| \leqslant |W_{x(Q)} \cap W_{x(Q')}| \leqslant c_{\mathrm{LRS}}(t),$$
where the first inequality follows from (r1), the second from the properties of tree decompositions, and the last from (lrs1). ◊

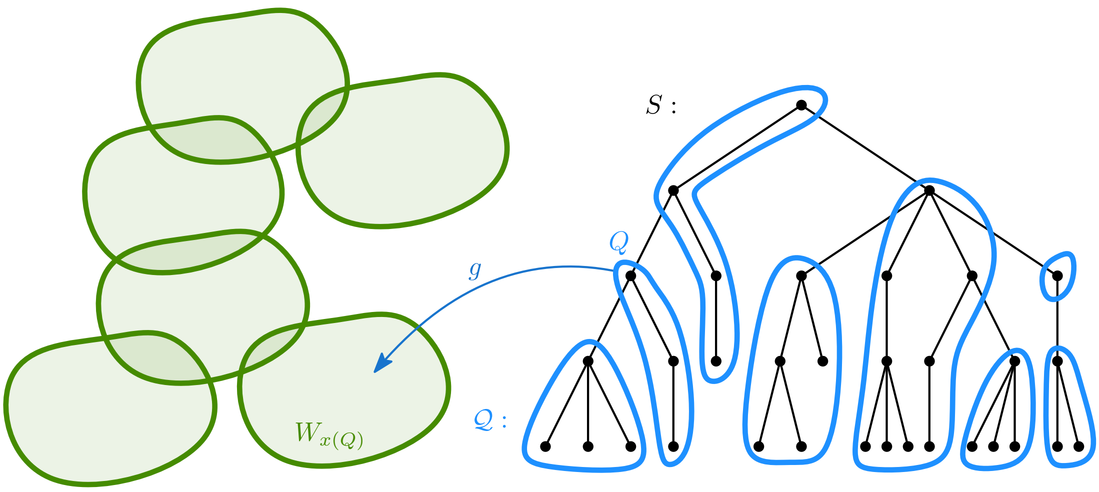


Figure 6.19: Intuitively, the part of the tree decomposition $\mathcal{V}$ corresponding to the vertices in Q comes from $\mathcal{D}_{x(Q)}$.

Claim 6.56.2. *Let $x \in V(T)$, $y \in V(S)$, and let $i \in \mathbb{N}$. Then*
$$|V_y \cap L_{x,i}| \leqslant c_{\mathrm{LRS}}(t).$$

Proof of the claim. Since $(\mathcal{V}, \mathcal{Q})$ refines $(\mathcal{U}, \mathcal{P})$ we have that $V_y \subseteq U_{f(y)}$. Let $f(y) = (x', z)$ where $x' \in V(T)$ and $z \in V(T_{x'})$. Thus, $U_{f(y)} = U_{(x',z)} = (D_{x',z} \cup A_{x'}) \cap V(G_0)$. Recall that $A_x \cap L_{x,i} = \emptyset$. Thus in the case of $x = x'$,
$$|V_y \cap L_{x,i}| \leqslant |U_{(x,z)} \cap L_{x,i}| \leqslant |(D_{x,z} \cup A_x) \cap L_{x,i}| \leqslant |D_{x,z} \cap L_{x,i}| \leqslant c_{\mathrm{LRS}}(t),$$
where the last inequality follows by (lrs5). Finally, in the case of $x \neq x'$, we have $U_{(x',z)} \subseteq W_{x'}$ and $L_{x,i} \subseteq W_x$, and so,
$$|V_y \cap L_{x,i}| \leqslant |U_{(x',z)} \cap L_{x,i}| \leqslant |W_{x'} \cap W_x| \leqslant c_{\mathrm{LRS}}(t),$$

where the last inequality follows from (lrs1). ◊

We root S in an arbitrary vertex $\mathrm{root}(S)$. For every $X \subseteq V(S)$, we define

$$\mathcal{Q}(X) = \{Q \in \mathcal{Q} \mid V(Q) \cap X \neq \emptyset\}.$$

Let d be a positive integer and let $\mathcal{F}$ be a family of connected subgraphs of G_0 such that there are no $d+1$ pairwise disjoint members of $\mathcal{F}$. In the remainder of the proof, we construct Z satisfying (qc1)-(qc3).

By Lemma 1.17, there exists $X_0 \subseteq V(S)$ of size at most d such that $\bigcup_{y \in X_0} V_y$ intersects every member of $\mathcal{F}$. Let

$$X_1 = \mathrm{LCA}(S, X_0).$$

By Lemma 5.2,

$$|\mathcal{Q}(X_1)| \leqslant |X_1| \leqslant 2d - 1. \tag{6.1}$$

The set $\bigcup_{y \in X_1} V_y$ is a hitting set of $\mathcal{F}$, thus, it is a good candidate to be Z. However, we still need to define ψ and the ultimate goal is to repeat the idea of coloring that we applied in the bounded layered treewidth case. Since $\mathcal{L}_x$ is a layering of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$, we need to take into account the vertices in A_x when building the final Z. Moreover, the coloring φ may not be compatible with layerings on the adhesions of $\mathcal{W}$, we also need to consider some of them. To this end, we will add some more vertices to Z, which we ultimately color injectively with a separate palette of colors. Because we work with $\mathcal{V}$ instead of $\mathcal{U}$ we need the following notions of projections. See also Figure 6.20 for an illustration.

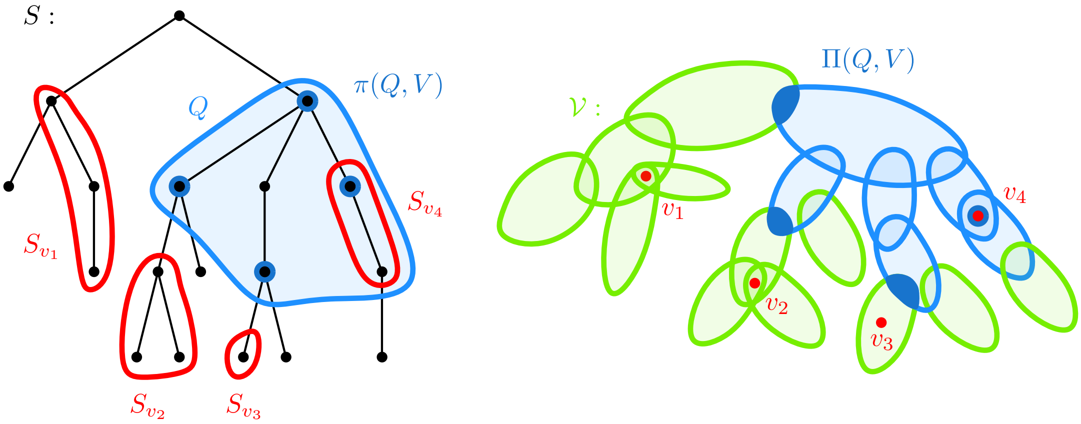


Figure 6.20: Here, $V = \{v_1, v_2, v_3, v_4\}$. Note that $\bigcup_{y \in V(Q)} V_y$ is separated from V by $\Pi(Q, V)$ in G_0 as we show later in Claim 6.56.3.

Let $Q \in \mathcal{Q}$ and $v \in V(G_0)$. Let $S_v = S[\{z \in V(S) \mid v \in V_z\}]$. Thus, Q and S_v are two subtrees of S. We define projections $\pi(Q, v)$ and $\Pi(Q, v)$ as follows. If Q and S_v share a vertex, then choose such a vertex z arbitrarily and set $\pi(Q, v) = z$ and $\Pi(Q, v) = \{v\}$. If $V(Q)$ and $V(S_v)$ are disjoint, then consider the shortest $(V(Q), V(S_v))$-path path in S. Let z be the endpoint of that path in Q and let y be the vertex adjacent to z in that path. Then set $\pi(Q, v) = z$ and $\Pi(Q, v) = V_y \cap V_z$. The definition can be naturally extended to subsets of $V(G_0)$. For each $V \subseteq V(G_0)$, let

$$\pi(Q, V) = \{\pi(Q, v) \mid v \in V\} \text{ and } \Pi(Q, V) = \bigcup_{v \in V} \Pi(Q, v).$$

Note that by Claim 6.56.1,

$$|\Pi(Q,V)| \leqslant c_{\mathrm{LRS}}(t) \cdot |V|. \tag{6.2}$$

The key property of these objects is the following. See Figure 6.20 again.

Claim 6.56.3. *Let $Q \in \mathcal{Q}$ and let $V \subseteq V(G_0)$. Every connected subgraph H of $G_0 - \Pi(Q,V)$ that intersects $\bigcup_{y\in V(Q)} V_y$ is disjoint from V.*

Proof. It suffices to observe that by the properties of tree decompositions and construction, $\Pi(Q,V)$ intersects every path between $\bigcup_{y\in V(Q)} V_y$ and V in G_0. Thus, if a connected subgraph H of G_0 has a vertex is both $\bigcup_{y\in V(Q)} V_y$ and V, then it has a vertex in $\Pi(Q,V)$. □

Let

$$Y_Q = \begin{cases} \pi(Q, A_{x(Q)}) & \text{if } x(Q) = r,\\ \pi(Q, A_{x(Q)} \cup (W_{x(Q)} \cap W_{\mathrm{p}(T,x(Q))})) & \text{if } x(Q) \neq r. \end{cases}$$

Recall that $|A_{x(Q)}| \leqslant c_{\mathrm{LRS}}(t)$ by (lrs2), and if $x(Q) \neq r$, then $|W_{x(Q)} \cap W_{\mathrm{p}(T,x(Q))}| \leqslant c_{\mathrm{LRS}}(t)$ by (lrs1). Thus, we have

$$\begin{aligned} |Y_Q| &\leqslant |A_{x(Q)}| && \leqslant c_{\mathrm{LRS}}(t) && \text{if } x(Q) = r,\\ |Y_Q| &\leqslant |A_{x(Q)}| + |W_{x(Q)} \cap W_{\mathrm{p}(T,x(Q))}| && \leqslant 2c_{\mathrm{LRS}}(t) && \text{if } x(Q) \neq r. \end{aligned} \tag{6.3}$$

Moreover, by definition $Y_Q \subseteq V(Q)$.

Let

$$X_2 = X_1 \cup \{\mathrm{root}(Q) \mid Q \in \mathcal{Q}(X_1)\} \cup \bigcup\{Y_Q \mid Q \in \mathcal{Q}(X_1)\}$$

and

$$X_3 = \mathrm{LCA}(S, X_2).$$

Observe that

$$\begin{aligned} |X_3| &\leqslant 2|X_2| - 1 && \text{by Lemma 5.2}\\ &\leqslant 2(|X_1| + |\mathcal{Q}(X_1)| + |\mathcal{Q}(X_1)| \cdot 2c_{\mathrm{LRS}}(t)) - 1 && \text{by (6.3)}\\ &\leqslant 2((2d-1) + (2d-1) + (2d-1) \cdot 2c_{\mathrm{LRS}}(t)) - 1 && \text{by (6.1)}\\ &\leqslant (8c_{\mathrm{LRS}}(t) + 8)d. \end{aligned} \tag{6.4}$$

Recall that by Lemma 6.50, for all $X, Y \subseteq V(S)$ with $X \subseteq Y$ and $\mathrm{LCA}(S,X) = X$, if $\mathcal{Q}(X) = \mathcal{Q}(Y)$, then $\mathcal{Q}(Y) = \mathcal{Q}(\mathrm{LCA}(S,Y))$. Since by construction $\mathcal{Q}(X_1) = \mathcal{Q}(X_2)$, we can apply the above with $X = X_1$ and $Y = X_2$ to obtain that

$$\mathcal{Q}(X_1) = \mathcal{Q}(X_2) = \mathcal{Q}(X_3).$$

Finally, the set Z that witnesses the assertion of the claim is given by

$$Z = \bigcup_{y\in X_3} V_y.$$

Next, consider $Q \in \mathcal{Q}(X_1)$ and let

$$B_Q = \begin{cases} \Pi(Q, A_{x(Q)}) & \text{if } x(Q) = r,\\ \Pi(Q, A_{x(Q)} \cup (W_{x(Q)} \cap W_{\mathrm{p}(T,x(Q))})) & \text{if } x(Q) \neq r. \end{cases}$$

Since for every $v \in V(G_0)$, $\Pi(Q,v) \subseteq V_{\pi(Q,v)}$ and $Y_Q \subseteq X_2 \subseteq X_3$, we have

$$B_Q \subseteq \bigcup_{y \in Y_Q} V_y \subseteq \bigcup_{y \in X_2} V_y \subseteq Z.$$

Recall that $|A_{x(Q)}| \leqslant c_{\mathrm{LRS}}(t)$ by (lrs2), and if $x(Q) \neq r$, then $|W_{x(Q)} \cap W_{\mathrm{p}(T,x(Q))}| \leqslant c_{\mathrm{LRS}}(t)$ by (lrs1). Thus, by (6.2), we have

$$\begin{aligned} |B_Q| &\leqslant |A_{x(Q)}| \cdot c_{\mathrm{LRS}}(t) && \leqslant (c_{\mathrm{LRS}}(t))^2 && \text{if } x(Q) = r, \\ |B_Q| &\leqslant \left|A_{x(Q)} \cup (W_{x(Q)} \cap W_{\mathrm{p}(T,x(Q))}))\right| \cdot c_{\mathrm{LRS}}(t) && \leqslant 2(c_{\mathrm{LRS}}(t))^2 && \text{if } x(Q) \neq r. \end{aligned} \tag{6.5}$$

Let

$$B = \left(\bigcup_{Q \in \mathcal{Q}(X_1), \mathrm{root}(S) \notin V(Q)} V_{\mathrm{root}(Q)} \cap V_{\mathrm{p}(S,\mathrm{root}(Q))} \right) \cup \left(\bigcup_{Q \in \mathcal{Q}(X_1)} B_Q \right).$$

Recall that for every $Q \in \mathcal{Q}(X_1)$ with $\mathrm{root}(S) \notin V(Q)$, $\mathrm{root}(Q) \in X_2$ and so $V_{\mathrm{root}(Q)} \cap V_{\mathrm{p}(S,\mathrm{root}(Q))} \subseteq V_{\mathrm{root}(Q)} \subseteq Z$. Moreover, for every $Q \in \mathcal{Q}(X_1)$, $B_Q \subseteq Z$. Therefore,

$$B \subseteq Z.$$

Observe also that

$$\begin{aligned} |B| &\leqslant |\mathcal{Q}(X_1)| \cdot c_{\mathrm{LRS}}(t) + |\mathcal{Q}(X_1)| \cdot 2(c_{\mathrm{LRS}}(t))^2 && \text{by Claim 6.56.1 and (6.5)} \\ &\leqslant (2d-1) \cdot c_{\mathrm{LRS}}(t) + (2d-1) \cdot 2(c_{\mathrm{LRS}}(t))^2 && \text{by (6.1)} \\ &\leqslant c_{\mathrm{LRS}}(t)(4c_{\mathrm{LRS}}(t) + 2)d. \end{aligned} \tag{6.6}$$

Claim 6.56.4. *For all distinct $Q_1, Q_2 \in \mathcal{Q}(X_1)$, there is no path between $\bigcup_{y \in V(Q_1)} V_y \setminus B$ and $\bigcup_{y \in V(Q_2)} V_y \setminus B$ in $G_0 - B$. In particular, $\bigcup_{y \in V(Q_1)} V_y \setminus B$ and $\bigcup_{y \in V(Q_2)} V_y \setminus B$ are disjoint.*

Proof of the claim. Let Q_1, Q_2 be distinct members of $\mathcal{Q}(X_1)$. Recall that Q_1, Q_2 are disjoint subtrees of S. Note that there exists $i \in \{1,2\}$ such that $\mathrm{root}(Q_i) \neq \mathrm{root}(S)$ and every path in S between Q_1 and Q_2 goes through the edge $\mathrm{root}(Q_i)\,\mathrm{p}(S,\mathrm{root}(Q_i))$ of S. Hence by properties of tree decompositions, $V_{\mathrm{root}(Q_i)} \cap V_{\mathrm{p}(S,\mathrm{root}(Q_i))}$ intersects every path between $\bigcup_{y \in V(Q_1)} V_y$ and $\bigcup_{y \in V(Q_2)} V_y$ in G_0. Since $V_{\mathrm{root}(Q_i)} \cap V_{\mathrm{p}(S,\mathrm{root}(Q_i))} \subseteq B$, this proves the claim. $\Diamond$

As a consequence of Claim 6.56.4, $\{(\bigcup_{y \in V(Q)} V_y) \cap (Z \setminus B) \mid Q \in \mathcal{Q}(X_1)\}$ is a family of pairwise disjoint sets covering $Z \setminus B$ (because $\mathcal{Q}(X_1) = \mathcal{Q}(X_3)$). We want to refine this family with the layerings of the torsos. To this end, we need the following fact where we substantially use the fact that $(\mathcal{V}, \mathcal{Q})$ refines $(\mathcal{U}, \mathcal{P})$.

Claim 6.56.5. *For every $Q \in \mathcal{Q}(X_1)$ and $V \subseteq V(G_0)$ such that $\Pi(Q,V) \subseteq B_Q$, we have*

$$\bigcup_{y \in V(Q)} V_y \setminus B \subseteq W_{x(Q)} \setminus V.$$

Proof of the claim. Let $Q \in \mathcal{Q}(X_1)$ and let $V \subseteq V(G_0)$ be such that $\Pi(Q,V) \subseteq B_Q$. Since $\Pi(Q,V) \subseteq B_Q \subseteq B$, then

$$\bigcup_{y\in V(Q)} V_y \setminus B \subseteq \bigcup_{y\in V(Q)} V_y \setminus B_Q \subseteq \bigcup_{y\in V(Q)} V_y \setminus \Pi(Q,V).$$

By claim 6.56.3, $\bigcup_{y\in V(Q)} V_y \setminus \Pi(Q,V)$ is disjoint from V. Thus, to conclude the claim, it suffices to show that $\bigcup_{y\in V(Q)} V_y \subseteq W_{x(Q)}$. Recall that $g(Q) = T_0[\{x(Q)\} \times T_{x(Q)}]$. We have

$$\bigcup_{y\in V(Q)} V_y \subseteq \bigcup_{y\in V(Q)} U_{f(y)} \subseteq \bigcup_{z\in V(g(Q))} U_z = \bigcup_{z\in\{x(Q)\}\times T_{x(Q)}} U_z \subseteq W_{x(Q)}$$

where the first inclusion follows from (r1), the second from (r2), and the last one from the construction of $\mathcal{U}$. ♢

Since $\Pi(Q, A_{x(Q)}) \subseteq B_Q$ for every $Q \in \mathcal{Q}(X_1)$, by claim 6.56.5,

$$\left(\bigcup_{y\in V(Q)} V_y \setminus B\right) \cap A_{x(Q)} = \emptyset.$$

Recall that for every $Q \in \mathcal{Q}(X_1)$, $(L_{x(Q),i} \mid i \in \mathbb{N})$ is a layering of $\mathrm{torso}_{G,\mathcal{W}}(W_{x(Q)}) - A_{x(Q)}$. It follows that the family

$$\mathcal{B} = \left\{\left(\bigcup_{y\in V(Q)} V_y\right) \cap L_{x(Q),i} \cap (Z \setminus B) \mid Q \in \mathcal{Q}(X_1), i \in \mathbb{N}\right\}$$

is a family of pairwise disjoint sets covering $Z \setminus B$. Moreover, the members of $\mathcal{B}$ are of reasonable size, namely, for every $Q \in \mathcal{Q}(X_1)$ and $i \in \mathbb{N}$,

$$\begin{aligned}\left|\left(\bigcup_{y\in V(Q)} V_y\right) \cap L_{x(Q),i} \cap (Z \setminus B)\right| &\leqslant |L_{x(Q),i} \cap Z| \leqslant \sum_{y\in X_3} |L_{x(Q),i} \cap V_y| \\ &\leqslant \sum_{y\in X_3} |L_{x(Q),i} \cap U_{f(y)}| && \text{by (r1)} \\ &\leqslant c_{\mathrm{LRS}}(t) \cdot |X_3| && \text{by (lrs5)} \\ &\leqslant c_{\mathrm{LRS}}(t) \cdot (8c_{\mathrm{LRS}}(t) + 8)d && \text{by (6.4).}\end{aligned} \tag{6.7}$$

We define a coloring ψ of Z using at most $c_{6.56}(t)$ colors. See also Figure 6.21. First, we color B injectively, and then we color each member of $\mathcal{B}$ also injectively avoiding colors in $\psi(B)$. In the first step we used at most $c_{\mathrm{LRS}}(t)(4c_{\mathrm{LRS}}(t) + 2)d$ colors by (6.6) and in the second step, we used at most $c_{\mathrm{LRS}}(t) \cdot (8c_{\mathrm{LRS}}(t) + 8)d$ colors by (6.7), thus, ψ is well-defined.

We now show that Z satisfy (qc1)-(qc3).

Recall that X_0 was chosen so that $X_0 \subseteq X_3$ and $\bigcup_{y\in X_0} V_y$ intersects every member of $\mathcal{F}$. Therefore, (qc1) holds. Item (qc2) holds by Lemma 5.4 since $Z = \bigcup_{y\in \mathrm{LCA}(S,X_2)} V_y$, and $\mathcal{V} = (S, (V_y \mid y \in V(S)))$ is a natural tree decomposition of G_0.

It remains to prove (qc3). Let $P \subseteq Z$ and let $R = P \cup B$. First, $|R \setminus P| \leqslant |B| \leqslant c_{\mathrm{LRS}}(t)(4c_{\mathrm{LRS}}(t) + 2)d$. Consider a connected subgraph H of G_0 such that $V(H) \cap P \neq \emptyset$. If $V(H) \cap B \neq \emptyset$, then since every vertex of B has a unique color, any vertex in $V(H) \cap B$ is a $\psi|_R$-center of $V(H) \cap P$, and so, a $(\varphi \times \psi)$-center of $V(H) \cap Z$. Thus, we assume that $V(H) \cap B = \emptyset$.

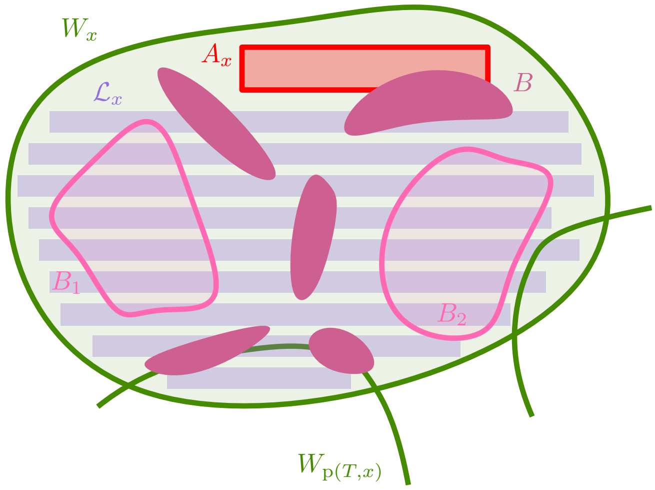


Figure 6.21: The set Z is the union of B and the members of $\mathcal{B}$. We color the vertices in B injectively. The role of the set B is to separate members of $\mathcal{B}$ so that if B is intersected by a connected subgraph H of G_0, we immediately get a center. In the figure, $B_1 = \bigcup_{y \in V(Q_1)} V_y \setminus B$ and $B_2 = \bigcup_{y \in V(Q_2)} V_y \setminus B$ for some $Q_1, Q_2 \in \mathcal{Q}(X_3)$. If a connected subgraph H of G_0, intersects both B_1 and B_2, it has to intersect B.

Since $V(H) \cap B = \emptyset$, it follows from Claim 6.56.4, that $V(H) \cap Z \subseteq \bigcup_{z \in V(Q)} V_z$ for some $Q \in \mathcal{Q}(X_1)$.

Additionally, by Claim 6.56.5,

$$\begin{aligned} V(H) \cap Z &\subseteq W_x \setminus A_x && \text{if } x = r, \\ V(H) \cap Z &\subseteq W_x \setminus (A_x \cup W_{\mathrm{p}(T,x)}) && \text{if } x \neq r. \end{aligned}$$

Recall that $(L_{x,i} \mid i \in \mathbb{N})$ is a layering of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$. Moreover, by definition of φ, for every $u \in V(H) \cap Z$, $\varphi(u) = i \bmod (q+1)$ where $i \in \mathbb{N}$ is such that $u \in L_{x,i}$.

Consider $u \in V(H) \cap R$. Let $i \in \mathbb{N}$ be such that $u \in L_{x,i}$. If u is a $(\varphi \times \psi)$-center in $V(H) \cap R$, then we are done. Assume now that there exists $u' \in V(H) \cap R$ distinct from u such that $\varphi(u) = \varphi(u')$ and $\psi(u) = \psi(u')$. Let $j \in \mathbb{N}$ be such that $u' \in L_{x,j}$. Without loss of generality, assume that $i \leqslant j$. By the definition of ψ, $j \neq i$, and by the definition of φ, $|j - i| > q$. By Lemma 6.51, $V(H) \cap W_x$ induces a connected subgraph of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$. Since $(L_{x,k} \mid k \in \mathbb{N})$ is a layering of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$, it follows that $V(H)$ intersects $L_{x,k}$ for every $k \in \{i, \dots, j-1\}$. We deduce that $|\varphi(V(H))| > q$. This proves that $\psi|_{V(H)}$ witnesses the fact that

$$\mathrm{cen}_q(G, \varphi, R) \leqslant (c_{\mathrm{LRS}}(t)(4c_{\mathrm{LRS}}(t) + 2) + c_{\mathrm{LRS}}(t)(8c_{\mathrm{LRS}}(t) + 8)) \cdot d$$

Therefore, we obtain (qc3), which ends the proof. □

6.9.4 Excluding an apex-forest

In this section, we prove what will be the base case in the proof of Theorem 1.34, which corresponds to excluding an apex-forest. The core of the proof is inspired by a result of Dębski, Felsner, Micek, and Schröder [DMSF21] which asserts that outer-planar graphs have q-centered chromatic number in $\mathcal{O}(q \log q)$. See Figure 6.22 for a sketch of their argument. Since outer-planar graphs exclude an apex-forest, namely $K_{2,3}$, this section can be seen as a generalization of their result. As in Section 6.6, we also use ideas from Dujmović, Hickingbotham, Joret, Micek, Morin, and Wood's paper [DHJ+23]. We start by proving structural properties for graphs with no $\mathcal{F}$-rich model of a given star. Then, we will apply it inductively to extend it to graphs with no $\mathcal{F}$-rich model of a given forest, and finally graphs with no $\mathcal{F}$-rich model of a given apex-forest. Recall that for all positive integer h, d, we denote by $F_{h,d}$ the complete d-ary tree of vertex-height h. In particular, $F_{2,d}$ is the star with d leaves.

Lemma 6.58. *Let q, c, d be positive integers, let G be a connected graph, let φ be a (q, c)-good coloring of G, let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{2,d}$, and let $U \subseteq V(G)$ such that $G[U]$ is connected. There exists $S \subseteq V(G)$ and a path partition $(P_0, \dots, P_\ell)$ of (G, S) and sets $R_1, \dots, R_\ell \subseteq V(G)$ such that*

(a) *$P_0 = U$;*

(b) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;*

(c) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most three connected components of $G - V(C)$;*

(d) *$P_i \subseteq R_i$ for every $i \in [\ell]$;*

(e) *$\mathrm{cen}_q(G, \varphi, R_i) \leqslant c \cdot d$ for every $i \in \{2, \dots, \ell\}$; and*

(f) *$|R_i \setminus P_i| \leqslant c \cdot d$ for every $i \in \{2, \dots, \ell\}$.*

Proof. We proceed by induction on $|V(G) \setminus U|$. If every member of $\mathcal{F}$ intersects U, then the result holds for $\ell = 0$ and $P_0 = U$. Now suppose that $\mathcal{F}|_{G-U} \neq \emptyset$ (and so $V(G) \setminus U \neq \emptyset$), and that the result holds for smaller values of $|V(G) \setminus U|$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of $G - U$ such that $N_G(V(H)) \cap U \neq \emptyset$ and there exists $F \in \mathcal{F}$ such that $F \subseteq H$.

We claim that there are no $d + 1$ pairwise disjoint members of $\mathcal{F}'$. Indeed, if $H_1, \dots, H_{d+1}$ are $d+1$ pairwise disjoint members of $\mathcal{F}'$, then $(V(H_1), \dots, V(H_d), U \cup V(H_{d+1}))$ is an $\mathcal{F}$-rich model of $F_{2,d}$ in G, a contradiction. Hence, since φ is a strongly (q, c)-good coloring of G, there exists $Z \subseteq V(G)$ such that

(qc1') $V(F) \cap Z \neq \emptyset$ for every $F \in \mathcal{F}'$;

(qc2') for every connected component C of $G - Z$, $N_G(V(C))$ intersects at most two connected components of $G - V(C)$; and

(qc3') for every $P \subseteq Z$, there exists $R \subseteq Z$ with

- (a) $P \subseteq R$,
- (b) $\mathrm{cen}_q(G, \varphi, R) \leqslant c \cdot d$, and
- (c) $|R \setminus P| \leqslant c \cdot d$.

Let P be an inclusion wise minimal subset of $Z \setminus U$ satisfying the following properties:

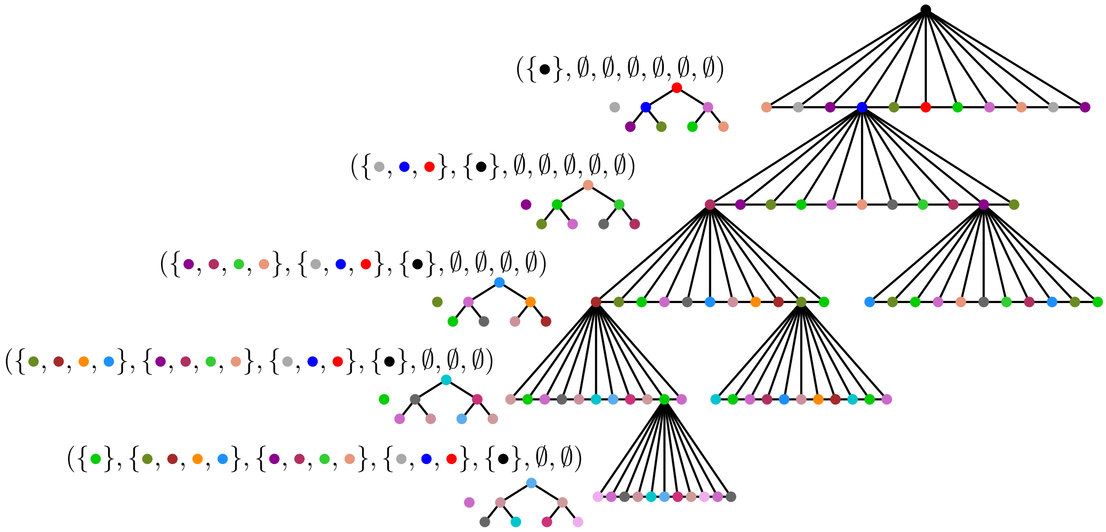


Figure 6.22: Sketch of Dębski, Felsner, Micek, and Schröder's argument in [DMSF21] to prove that outer-planar graphs have q-centered chromatic number in $\mathcal{O}(q \log q)$. Assume $q = 2^k - 1$ for some $k \in \mathbb{N}$ ($k = 3$ and $q = 7 = 2^3 - 1$ in the picture). For simplicity, we consider a graph G which is a "tree of fans". Let Y be a set of $q(k+2)+1 = \mathcal{O}(q \log q)$ colors. We will construct from top to bottom a q-centered coloring of G using only colors in Y. For the root, we chose an arbitrary color in Y. Now, to color a subtree with root path P, we will have some forbidden colors because of the above layers. This is encoded as a sequence $(Y_1, \ldots, Y_q)$ of subsets of Y, each of size at most $k+1$. The set Y_a for $a \in [q]$ represents the colors forbidden because of the a-th previous layer. To color the current path P, we use $q+1$ colors in $Y \setminus \bigcup_{a \in [q]} Y_a$. Among these colors, we chose one to be special, and we arrange the $q = 2^k - 1$ other ones in a complete binary tree. Then, we color P periodically with a pattern starting with the special color, followed by the q colors in the auxiliary tree in a post-order. This special color and this tree are depicted next some paths (we ignore the other paths due to lack of space). Once P is colored, it remains to color the subgraphs under P. For a vertex $u \in V(P)$, we apply the induction hypothesis to the subgraph under u for the new buffer $(\tilde{Y}_u, Y_1, \ldots, Y_{q-1})$, where $\tilde{Y}_u$ contains the special color and the ancestors of the color of u in the auxiliary tree. It remains to show that the obtained coloring ψ is q-centered. Let H be a connected subgraph of G using at most q colors, and let P be the highest path P in this tree structure intersecting $V(H)$. Then $V(P) \cap V(H)$ induces a subpath of P. If $V(H) \cap V(P)$ contains a vertex with the special color, then this vertex is a ψ-center of $V(H)$, since $V(H)$ uses at most q colors, and because the color of u is forbidden in every subtree below P in the next q layers. Otherwise, let $u \in V(H) \cap V(P)$ be such that the color $\psi(u)$ is the highest in the auxiliary tree. This is well defined because the set of colors used by $V(H) \cap V(P)$ as a unique element closest to the root. By construction, the color of u is forbidden in the next q layers of each subtree below P intersecting $V(H)$, and so u is a ψ-center of $V(H)$.

(i) $V(F) \cap P \neq \emptyset$ for every $F \in \mathcal{F}'$; and

(ii) for every connected component C of $G-U-P$ such that $\mathcal{F}|_C = \emptyset$, $N_{G-U}(V(C))$ intersects at most three connected components of $G - U - V(C)$.

Note that $Z \setminus U$ satisfies these two items and so P is well-defined.

Let W be the union of the vertex sets of all the connected components C of $G - U - P$ such that $\mathcal{F}|_C \neq \emptyset$. Note that $N_G(W) \cap U = \emptyset$.

We claim that for every $u \in P \cap N_G(W)$, there is a path from u to U in G disjoint from W. Indeed, by minimality of P, either there exists $F \in \mathcal{F}'$ disjoint from $P \setminus \{u\}$, or there is a connected component C of $G - U - (P \setminus \{u\})$ with $\mathcal{F}|_C = \emptyset$ such that $N_{G-U}(V(C))$ intersects at least four connected components of $G - U - V(C)$. In the first case, this member F of $\mathcal{F}'$ contains u and so there is a path from $N_G(U)$ to u in F. However, since $N_G(W) \cap U = \emptyset$, this path is disjoint from W and we are done. In the second case, there is a connected component C of $G - U - (P \setminus \{u\})$ with $\mathcal{F}|_C = \emptyset$ such that $N_{G-U}(V(C))$ intersects at least three connected components of $G - U - V(C)$. Since P satisfies (ii), C contains u. As $u \in N_G(W)$, it follows that there is a connected component C' of $G - U - P$ with $\mathcal{F}|_{C'} \neq \emptyset$ and $C' \subseteq C$. But then $\mathcal{F}|_C \neq \emptyset$, a contradiction.

Because G and $G[U]$ are connected, this implies that $U' = V(G) \setminus W$ induces a connected subgraph of G. Moreover, since $\mathcal{F}|_{G-U} \neq \emptyset$, the family $\mathcal{F}'$ is nonempty and so P is non-empty. This implies that $|V(G) \setminus U'| < |V(G) \setminus U|$. Hence, by the induction hypothesis, there exists $S' \subseteq V(G)$ and a path partition $(P'_0, \ldots, P'_{\ell'})$ of (G, S') and sets $R'_1, \ldots, R'_{\ell'} \subseteq V(G)$ such that

(a') $P'_0 = U'$;

(b') $V(F) \cap S' \neq \emptyset$ for every $F \in \mathcal{F}'$;

(c') for every connected component C of $G - S'$, $N_G(V(C))$ intersects at most three connected components of $G - V(C)$;

(d') $P'_i \subseteq R'_i$ for every $i \in [\ell']$;

(e') $\mathrm{cen}_q(G, \varphi, R'_i) \leqslant c \cdot d$ for every $i \in \{2, \ldots, \ell'\}$; and

(f') $|R'_i \setminus P'_i| \leqslant c \cdot d$ for every $i \in \{2, \ldots, \ell'\}$.

By (qc3'), there exists $R \subseteq Z$ with $P \subseteq R$, $\mathrm{cen}_q(G, \varphi, R) \leqslant c \cdot d$, and $|R \setminus P| \leqslant c \cdot d$. We set $\ell = \ell' + 1$, and

$$\begin{aligned} &P_0 = U, && \\ &P_1 = P, && R_1 = R, \\ &P_i = P'_{i-1}, && R_i = R'_{i-1} \quad \text{for every } i \in \{2, \ldots, \ell\}. \end{aligned}$$

Finally, let $S = \bigcup_{i \in [\ell]} P_i$.

By construction, (a) holds. Moreover, for every $F \in \mathcal{F}$, either $V(F) \cap P \neq \emptyset$ or $V(F) \cap (P'_1 \cup \cdots \cup P'_{\ell'}) \neq \emptyset$ by (b'). In both cases, $V(F) \cap S \neq \emptyset$ and so (b) holds.

Let C be a connected component of $G - S$. If $V(C) \cap U' = \emptyset$, then C is a connected component of $G - S'$ and so by (c'), $N_G(V(C))$ intersects at most three connected components of $G - V(C)$. Otherwise, C is a connected component of $G - U - P$ which does not contain any member of $\mathcal{F}$. Hence, by (ii), $N_{G-U}(V(C))$ intersects at most two connected components of $G - U - V(C)$. Since $G[U]$ is connected, we deduce that $N_G(V(C))$ intersects at most three connected components of $G - V(C)$. This proves (c).

By construction and (d'), (d) holds. By (e') and since $\mathrm{cen}_q(G, \varphi, R) \leqslant c \cdot d$, (e) holds. Finally, by (f') and because $|R \setminus P| \leqslant c \cdot d$, (f) holds. This proves the lemma. □

Lemma 6.59. *Let q, c, d be positive integers with $q \geqslant 2$ and let h be a nonnegative integer, let G be a graph, let φ be a (q, c)-good coloring of G, let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{h+1,d}$. There exists a set Y of at most $48c\left(hd + \binom{h}{2}\right) \cdot q \log q$ colors, a set $S \subseteq V(G)$, and a function $\lambda\colon S \to Y$ such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;*

(b) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most $8c\left(hd + \binom{h}{2}\right) \cdot \log q$ connected components of $G - V(C)$;*

(c) *there exists a family of subsets $Y_C \subseteq Y$ for C connected components of $G - S$, each of size at most $48c\left(dh + \binom{h}{2}\right) \cdot \log q$, such that for every connected subgraph H of G intersecting S, if $C_1, \dots, C_m$ are connected components of $G - S$ intersecting $V(H)$, then*

   (a) *$|\varphi(V(H))| > q$, or*

   (b) *$|\lambda(V(H) \cap S)| > q$, or*

   (c) *there is a $(\lambda \times \varphi)$-center u of $V(H) \cap S$ with $\lambda(u) \in Y_{C_i}$ for every $i \in [m]$.*

Proof. We proceed by induction on $h + |V(G)|$. When $h = 0$, $F_{h+1,d} = K_1$ and so $\mathcal{F}$ is empty. Hence $S = \emptyset$ satisfies the outcome of the lemma. Now suppose $h > 0$ and that the result holds for $h - 1$.

First assume that G is not connected. Let $\mathcal{C}$ be the family of all the connected components of G. By the induction hypothesis, and by possibly relabeling the obtained color sets, there exists a set Y of at most $48c\left(hd + \binom{h}{2}\right) \cdot q \log q$ colors, such that for every $C \in \mathcal{C}$, there exists a set $S_C \subseteq V(C)$ and a function $\lambda_C\colon S_C \to Y_C$ such that

(a') $V(F) \cap S_C \neq \emptyset$ for every $F \in \mathcal{F}|_C$;

(b') for every connected component C' of $C - S_C$, $N_C(V(C'))$ intersects at most $8c\left(hd + \binom{h}{2}\right) \cdot \log q$ connected components of $C - V(C')$;

(c') there exists a family of subsets $Y_{C'} \subseteq Y$ for C' connected component of $C - S_C$, each of size at most $48c\left(dh + \binom{h}{2}\right) \cdot \log q$, such that for every connected subgraph H of C intersecting S_C, if $C_1, \dots, C_m$ are connected components of $C - S_C$ intersecting $V(H)$, then

   (a) $|\varphi(V(H))| > q$, or

   (b) $|\lambda_C(V(H) \cap S_C)| > q$, or

   (c) there is a $(\lambda_C \times \varphi)$-center u of $V(H) \cap S_C$ with $\lambda_C(u) \in Y_{C_i}$ for every $i \in [m]$.

Let

$$S = \bigcup_{C \in \mathcal{C}} S_C,$$

and let $\lambda\colon S \to Y$ such that $\lambda|_{S_C} = \lambda_C$ for every $C \in \mathcal{C}$. Then, $S, \lambda, (Y_{C'} \mid C'$ connected component of $G - S)$ satisfies the outcome of the lemma. Now suppose that G is connected.

Let $\mathcal{F}'$ be the family of all the connected subgraphs of G containing an $\mathcal{F}$-rich model of $F_{h,d+1}$. We claim that there are no $\mathcal{F}'$-rich model of $F_{2,d}$ in G.

Suppose for a contradiction that there is an $\mathcal{F}'$-rich model $(B_x \mid x \in V(F_{2,d}))$ of $F_{2,d}$ in G. We will deduce an $\mathcal{F}$-rich model of $F_{h,d}$ in G. We denote by c the center of the star $F_{2,d}$ and by $x_1, \dots, x_d$ the leaves of $F_{2,d}$. Let $i \in [d]$. Since $(B_x \mid x \in V(F_{2,d}))$ is $\mathcal{F}'$-rich, B_{x_i} contains a neighbor u_i of B_c, and $G[B_{x_i}]$ has an $\mathcal{F}$-rich model $(B^i_x \mid x \in V(F_{h-1,d+1}))$ of $F_{h-1,d+1}$. By taking the sets $(B^i_x \mid x \in V(F_{h-1,d+1}))$ inclusion wise maximal, and because $G[B_{x_i}]$ is connected, we can assume that $u_i \in B^i_{y_i}$ for some $y_i \in F_{h-1,d+1}$. By Lemma 3.16, there is an $\mathcal{F}$-rich model $\mathcal{M}_i$ of $F_{h-1,d}$ in $G[B_{x_i}]$ whose branch set of the root contains a neighbor of B_c. It follows that the union of the models $\mathcal{M}_1, \dots, \mathcal{M}_d$, together with B_c as the branch set of the root, yields an $\mathcal{F}$-rich model of $F_{h,d}$ in G, a contradiction. This proves that there is no $\mathcal{F}'$-rich model of $F_{2,d}$ in G.

Hence by Lemma 6.58, applied for U being an arbitrary singleton, there exists $S' \subseteq V(G)$ and a path partition $(P_0, \dots, P_\ell)$ of (G, S') and sets $R_1, \dots, R_\ell \subseteq V(G)$ such that

6.58.(a) $P_0 = U$;

6.58.(b) $V(F) \cap S' \neq \emptyset$ for every $F \in \mathcal{F}'$;

6.58.(c) for every connected component C of $G - S'$, $N_G(V(C))$ intersects at most three connected components of $G - V(C)$;

6.58.(d) $P_i \subseteq R_i$ for every $i \in [\ell]$;

6.58.(e) $\mathrm{cen}_q(G, \varphi, R_i) \leqslant c \cdot d$ for every $i \in \{2, \dots, \ell\}$; and

6.58.(f) $|R_i \setminus P_i| \leqslant c \cdot d$.

We set $R_0 = P_0 = U$. Note that 6.58.(d)-6.58.(f) also holds for $i = 0$.

Let $i \in \{0, \dots, \ell\}$. Let $\eta_i\colon R_i \to [c \cdot d]$ witness 6.58.(e), and let $\psi_i\colon R_i \to [2c \cdot d]$ be such that $\psi_i|_{P_i} = \eta_i|_{P_i}$, $\psi_i|_{R_i \setminus P_i}$ is injective, and $\psi_i(u) \neq \psi_i(v)$ for every $u \in P_i$ and $v \in R_i \setminus P_i$. Such a coloring exists by 6.58.(f).

Let $\mathcal{C}$ be the family of all the connected components of $G - S'$. For every $C \in \mathcal{C}$, fix an index $i(C) \in [\ell]$ such that $N_G(V(C)) \subseteq P_{i(C)-1} \cup P_{i(C)}$. Now, for every $i \in [\ell]$, let

$$W_i = P_{i-1} \cup P_i \cup \bigcup_{C \in \mathcal{C}, i(C)=i} V(C).$$

Note that $(W_1, \dots, W_\ell)$ is a path decomposition of G.

For every $u \in V(G)$, let

$$\mu(u) = \begin{cases} i \bmod (q+1) & \text{if } u \in S', \text{ where } i \in \{0, \dots, \ell\} \text{ is such that } u \in P_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $s = \lceil \log(3q+2) \rceil$. For every $r \in \{0, \dots, s\}$, let $I_r = \{i \in \{0, \dots, \ell\} \mid i \equiv 0 \bmod 2^r\}$.

We will now build by induction on $s-r$ pairwise disjoint sets R'_i for $i \in I_r$. We will denote by S_r the set $\bigcup_{i\in I_r} R'_i$. We maintain the property that $P_i \subseteq \bigcup_{j\in I_r} R'_j$ for every $i \in I_r$.

First consider the base case $r=s$. For every $i \in I_s$, let

$$R'_i = R_i \cap \bigcup_{j\in\{0,\dots,\ell\}, i-q\leqslant j\leqslant i+q+1} W_j.$$

Since $W_i \cap W_{i+2} = \emptyset$ for every $i \in [\ell-2]$, and because $2^s \geqslant 3q+2$, the sets R'_i for $i \in I_s$ are pairwise disjoint. Let $S_s = \bigcup_{i\in I_s} R'_i$.

Now suppose $r < s$ and that R'_i for $i \in I_{r+1}$ has been constructed. For sake of presentation, we set $R'_i = P_i = \emptyset$ for every integer i with $i < 0$ or $i > \ell$. Let $i \in I_r \setminus I_{r+1}$. Note that $i - 2^r, i + 2^r \equiv 0 \bmod 2^{r+1}$. Let

$$R'_i = \left(R_i \cap \bigcup_{j\in\{0,\dots,\ell\}, i-2^r+1\leqslant j\leqslant i+2^r} W_i \right) \setminus S_{r+1}.$$

This complete the definition of the sets R'_i for $i \in \{0,\dots,\ell\}$. We define a coloring λ of S_0 as follows. For every $r \in \{0,\dots,s\}$, for every $i \in I_r \setminus I_{r+1}$ with the convention $I_{s+1} = \emptyset$, let

$$\lambda_0(u) = (r, \psi_i(u))$$

for every $u \in R'_i$.

Now that S_0 and λ_0 are defined, we decompose $G - S_0$. Since $S' = \bigcup_{i\in\{0,\dots,\ell\}} P_i$ intersects every member of $\mathcal{F}'$, and because $S' \subseteq S_0$, $G - S_0$ does not have any $\mathcal{F}$-rich minor of $F_{h,d+1}$. Hence, by the induction hypothesis, there exists a set $\tilde{Y}$ of at most $48c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot q \log q$ colors, a set $\tilde{S} \subseteq V(G - S_0)$ and a function $\tilde{\lambda}\colon \tilde{S} \to Y$ such that

(a”) $V(F) \cap \tilde{S} \neq \emptyset$ for every $F \in \mathcal{F}|_{G-S_0}$;

(b”) for every connected component C of $G - S_0 - \tilde{S}$, $N_{G-S_0}(V(C))$ intersects at most $9c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot \log q$ connected components of $G - S_0 - V(C)$;

(c”) there exists a family of subsets $\tilde{Y}_C \subseteq \tilde{Y}$ for C connected component of $G - S_0 - \tilde{S}$, each of size at most $48c\left((d+1)(h-1) + \binom{h-1}{2}\right) \cdot \log q$, such that for every connected subgraph H of $G - S_0$ intersecting $\tilde{S}$, if $C_1,\dots,C_m$ are connected components of $G - S_0$ intersecting $V(H)$, then

  (a) $|\varphi(V(H))| > q$, or
  (b) $|\tilde{\lambda}(V(H) \cap \tilde{S})| > q$, or
  (c) there is a $(\tilde{\lambda} \times \varphi)$-center u of $V(H) \cap \tilde{S}$ with $\tilde{\lambda}(u) \in \tilde{Y}_{C_i}$ for every $i \in [m]$.

Up to relabeling the elements of $\tilde{Y}$, we can assume that $\tilde{Y}$ is disjoint from the image of $\mu \times \lambda_0$.

Now, let

$$S = S_0 \cup \tilde{S},$$

and for every $u \in S$, let

$$\lambda(u) = \begin{cases} (\mu(u), \lambda_0(u)) & \text{if } u \in S_0, \\ \tilde{\lambda}(u) & \text{if } u \in \tilde{S}. \end{cases}$$

Let Y be the image of λ. First, note that

$$\begin{aligned}|Y| &\leqslant (q+1)\cdot(s+1)\cdot 2c(d+1) + 48c\left((h-1)(d+1)+\binom{h-1}{2}\right)\cdot q\log q\\ &\leqslant 48cd\cdot q\log q + 48c\left((h-1)(d+1)+\binom{h-1}{2}\right)\cdot q\log q\\ &\leqslant 48c\left(hd+\binom{h}{2}\right)\cdot q\log q.\end{aligned}$$

The main difficulty is now to prove (c). To do so, we use the following claim.

Claim 6.59.1. *There is a family of subsets $Y_i \subseteq Y$ for $i \in \{0,\dots,\ell\}$, each of size at most $48cd\cdot\log q$, such that for every connected subgraph H of G intersecting S_0, if $V(H)$ intersects $W_{i_1},\dots,W_{i_m}$ for $i_1,\dots,i_m \in \{0,\dots,\ell\}$, then*

(1) *$|\varphi(V(H))| > q$, or*

(2) *$|\lambda(V(H)\cap S_0)| > q$, or*

(3) *there is a $(\varphi\times\lambda)$-center u of $V(H)\cap S_0$ with $\lambda(u)\in Y_{i_j}$ for every $j\in[m]$.*

Proof of the claim. Let $i\in\{0,\dots,\ell\}$. For every $r\in\{0,\dots,s\}$, let i_r^- be the maximum element of I_r less or equal to i, or 0 if no such index exists, and let i_r^+ be the minimum element of I_r greater than i, or ℓ if no such index exists. Let

$$Y_i = \bigcup_{r\in\{0,\dots,s\},\varepsilon\in\{-,+\}} \left(\lambda(R'_{i_r^\varepsilon}\setminus P_{i_r^\varepsilon}) \cup (\{r\}\times\psi_{i_r^\varepsilon}(P_{i_r^\varepsilon}))\right).$$

Observe that

$$\begin{aligned}|Y_i| &\leqslant 2(s+1)\cdot(c(d+1)+c(d+1))\\ &\leqslant 2(2+\log(3q+2))\cdot 4cd\\ &\leqslant 8cd(2+\log(5q))\\ &\leqslant 8cd(5+\log q)\\ &\leqslant 48cd\cdot\log q.\end{aligned}$$

We now prove that Y_i for $i\in\{0,\dots,m\}$ is as desired. Let H be a connected subgraph of G intersecting S_0, and let $i_1,\dots,i_m\in\{0,\dots,\ell\}$ be such that $V(H)$ intersects W_{i_j} for each $j\in[m]$. If $|\varphi(V(H))|>q$ or $|\lambda(V(H)\cap S_0)|>q$ we are done. Now suppose $|\varphi(V(H))|\leqslant q$ and $|\lambda(V(H)\cap S_0)|\leqslant q$. This second inequality implies $|\mu(V(H)\cap S_0)|\leqslant q$, and so $V(H)$ intersects at most q parts of the path partition $(P_0,\dots,P_\ell)$. Let $r\in\{0,\dots,s\}$ be maximum such that $V(H)\cap S_r\neq\emptyset$.

First suppose that $r=s$. Since $V(H)$ intersects at most q parts of the path partition $(P_0,\dots,P_\ell)$, and because $s\geqslant 3q+2$, $V(H)$ intersects at most one of the sets R'_j for $j\in I_s$. Let $j\in I_s$ be this unique index. Then $j=i_s^\varepsilon$ for some $\varepsilon\in\{-,+\}$ and every $i\in\{i_1,\dots,i_m\}$. By the properties of R_j, there is a $(\varphi\times\eta_j)$-center u of $V(H)\cap R_j$. If $u\in R'_j$, then u is a $(\varphi\times\psi_j)$-center of $V(H)\cap R'_j$, and so u is a $(\varphi\times\lambda)$-center of $V(H)\cap R'_j$. This implies that u is a $(\varphi\times\lambda)$-center of $V(H)\cap S_0$, and $\lambda(u)\in\{s\}\times\psi_{i_s^\varepsilon}(P_{i_s^\varepsilon})\subseteq Y_i$ for every $i\in\{i_1,\dots,i_m\}$. Otherwise, if $u\notin R'_j$, then since $V(H)$ intersects at most q parts of the path partition $(P_0,\dots,P_\ell)$, $V(H)$ is disjoint from P_j. But then, since $\psi_j|_{R'_j\setminus P_j}$ is injective, any vertex v in $V(H)\cap R'_j$ is a $(\varphi\times\lambda)$-center of

$V(H) \cap S_0$, and $\lambda(v) \in \lambda(R'_{i_s^\varepsilon} \setminus P_{i_s^\varepsilon}) \subseteq Y_i$ for every $i \in \{i_1, \ldots, i_m\}$. In each case, we found a $(\varphi \times \lambda)$-center w of $V(H) \cap S_0$ with $\lambda(w) \in Y_i$ for every $i \in \{i_1, \ldots, i_m\}$.

Now assume $r < s$. In particular, $V(H)$ is disjoint from S_{r+1}. Since S_{r+1} separates every pair R'_{j_1}, R'_{j_2} for distinct $j_1, j_2 \in I_r \setminus I_{r+1}$, there is a unique $j \in I_r \setminus I_{r+1}$ such that $V(H) \cap R'_j \neq \emptyset$. Moreover, $j = i_r^\varepsilon$ for some $\varepsilon \in \{-, +\}$, for every $i \in \{i_1, \ldots, i_m\}$. Then, by the properties of R_j, there is a $(\varphi \times \eta_j)$-center u of $V(H) \cap R_j$. Since S_{r+1} separates R'_j from $R_j \setminus R'_j$, this vertex u belongs to R'_j. This implies that u is a $(\varphi \times \lambda)$-center of $V(H) \cap S_0$, and $\lambda(u) \in \{r\} \times \psi_{i_r^\varepsilon}(P_{i_r^\varepsilon}) \subseteq Y_i$ for every $i \in \{i_1, \ldots, i_m\}$. This proves the claim. $\diamondsuit$

We now show that (a)-(c) hold.

For every member F of $\mathcal{F}$, either $V(F) \cap S_0 \neq \emptyset$, or $V(F) \cap \tilde{S} \neq \emptyset$ by (a”), and so $V(F) \cap S \neq \emptyset$. This proves (a).

We now prove (b). Let C be a connected component of $G - S$. Let C' be the connected component of $G - S'$ containing C. First, by (b”), $N_{G-S_0}(V(C)) = N_G(V(C)) \cap \tilde{S}$ intersects at most $9c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot \log q$ connected component of $G - S_0 - V(C)$. Second, by 6.58.(c), $N_G(V(C'))$ intersects at most three connected components of $G - V(C')$. It is now enough to bound $|N_G(V(C)) \cap (S_0 \setminus S')|$. The set $N_G(V(C))$ intersects at most two of the R'_j for $j \in I_s$, and so, since $|R'_j \setminus P_j| \leqslant c \cdot d$ for every $j \in I_s$, we deduce

$$|N_G(V(C)) \cap (S_s \setminus S')| \leqslant 2c \cdot d.$$

Let $r \in \{0, \ldots, s-1\}$. Again, by construction, $N_G(V(C))$ intersects at most one of the R'_j for $j \in I_r \setminus I_{r+1}$, and so

$$|N_G(V(C)) \cap ((S_r \setminus S_{r+1}) \setminus S')| \leqslant c \cdot d.$$

Altogether, this yields

$$|N_G(V(C)) \cap (S_0 \setminus S')| \leqslant c \cdot d \cdot (s+2)$$

and so the number of connected components of $G - V(C)$ intersecting $N_G(V(C))$ is at most

$$\begin{aligned}
&3 + cd \cdot (s+2) + 9c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 3 + cd(3 + \log(2q+3)) + 9c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 3 + cd(3 + \log(5q)) + 9c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 3 + cd \cdot 6\log(q) + 9c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 9cd \cdot \log(q) + 9c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 9c\left(h(d+1) + \binom{h}{2}\right) \cdot \log q.
\end{aligned}$$

This proves (b).

Let C be a connected component of $G - S$. Let C' be the connected component of $G - S_0$ containing C, and let $\tilde{Y}_{C'} \subseteq \tilde{Y}$ be the set given by (c”). Let i be the unique index in $[\ell]$ such that W_i intersects $V(C)$. We set

$$Y_C = Y_i \cup \tilde{Y}_{C'}.$$

Note that

$$\begin{aligned}|Y_C| &\leqslant 24cd \cdot \log q + 24c\left((h-1)(d+1) + \binom{h-1}{2}\right) \cdot \log q \\ &\leqslant 24c\left(hd + \binom{h}{2}\right) \cdot \log q.\end{aligned}$$

We claim that the family Y_C for C connected component of $G - S$ satisfies (c).

Let H be a connected subgraph of G intersecting S, and let $C_1, \ldots, C_m$ be connected components of $G - S$ intersecting $V(H)$. For each $j \in [m]$, we denote by C'_j the connected component of $G - S_0$ containing C_j, and by i_j the unique integer in $[\ell]$ such that $V(C_j) \subseteq W_{i_j}$. If H is disjoint from S_0, then $C'_1 = \cdots = C'_m$, and by (c”), either

(i) $|\varphi(V(H))| > q$,

(ii) $|\tilde{\lambda}(V(H) \cap \tilde{S})| > q$ and so $\lambda(V(H) \cap S)| > q$, or

(iii) there is a $(\varphi \times \tilde{\lambda})$-center u of $V(H) \cap \tilde{S}$ with $\tilde{\lambda}(u) \in \tilde{Y}_{C'_1}$, and so u is a $(\varphi \times \lambda)$-center of $V(H) \cap S$ with $\lambda(u) \in Y_{C_i}$ for every $i \in [m]$.

Now suppose that $V(H)$ intersects S_0. Then by the claim, either

(i) $|\varphi(V(H))| > q$,

(ii) $|\lambda(V(H) \cap S)| > q$, or

(iii) there is a $(\varphi \times \lambda)$-center u of $V(H) \cap S_0$ with $\lambda(u) \in Y_i$ for every $i \in \{i_1, \ldots, i_m\}$. Because the colors in $\bigcup_{i \in \{i_1,\ldots,i_m\}} Y_i$ are used by λ only for vertices in S_0, we deduce that u is a $(\varphi \times \lambda)$-center of $V(H) \cap S$ with $\lambda(u) \in Y_{C_i}$ for every $i \in [m]$.

This shows (c), and concludes the proof of the lemma. □

The following lemma is folklore. See e.g. [Die17, Proposition 7.2.1] for a proof.

Lemma 6.60 (Folklore)**.** *Let k be an integer with $k \geqslant 3$. For every K_k-minor-free graph G, G as at most $2^{k-3} \cdot |V(G)| - 1$ edges.*

We now show a slightly modified version of Lemma 6.61, where in (b), the number of connected components of $G - V(C)$ intersecting $N_G(V(C))$ is now only a function of k, under the assumption that G is K_k-minor-free for some fixed positive integer k.

Lemma 6.61. *Let q, c, d, k be positive integers with $q \geqslant 2$ and $k \geqslant 3$. Let h be a nonnegative integer, let G be a K_k-minor-free graph, let φ be a (q, c)-good coloring of G, let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{h+1,d}$. There exists a set Y of at most $48c^2\left(hd + \binom{h}{2}\right) \cdot q \log q$ colors, a set $S \subseteq V(G)$, and a function $\lambda \colon S \to Y$ such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$;*

(b) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most 2^{k-1} connected components of $G - V(C)$;*

(c) *there exists a family $Y_C \subseteq Y$ for C connected component of $G - S$, each of size at most $56c^2\left(dh + \binom{h}{2}\right) \cdot \log q$, such that for every connected subgraph H of G intersecting S, if H intersects the connected components $C_1, \ldots, C_m$ of $G - S$, then*

(1) $|\varphi(V(H))| > q$, *or*

(2) $|\lambda(V(H) \cap S)| > q$, *or*

(3) *there is a* $(\lambda \times \varphi)$*-center* u *of* $V(H) \cap S$ *with* $\lambda(u) \in Y_{C_i}$ *for every* $i \in [m]$.

Proof. By Lemma 6.59, there exists a set Y_0 of at most $48c\left(hd + \binom{h}{2}\right) \cdot q \log q$ colors, a set $S_0 \subseteq V(G)$ and a function $\lambda_0 \colon S_0 \to Y_0$ such that

6.59.(a) $V(F) \cap S_0 \neq \emptyset$ for every $F \in \mathcal{F}$;

6.59.(b) for every connected component C of $G - S_0$, $N_G(V(C))$ intersects at most $8c\left(hd + \binom{h}{2}\right) \cdot \log q$ connected components of $G - V(C)$;

6.59.(c) there exists a family $Y_{0,C} \subseteq Y_0$ for C connected component of $G - S_0$, each of size at most $48c\left(dh + \binom{h}{2}\right) \cdot \log q$, such that for every connected subgraph H of G intersecting S, if $V(H)$ intersects the connected components $C_1, \ldots, C_m$ of $G - S_0$, then

(a) $|\varphi(V(H))| > q$, or

(b) $|\lambda_0(V(H) \cap S)| > q$, or

(c) there is a $(\varphi \times \lambda_0)$-center u of $V(H) \cap S_0$ with $\lambda_0(u) \in Y_{0,C_i}$ for every $i \in [m]$.

We assume that Y_0 is disjoint from $\mathbb{N}$.

Let $\mathcal{C}$ be the family of all the connected components of $G - S_0$. Let $C \in \mathcal{C}$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of C such that $N_G(V(H))$ intersects at least 2^{k-2} connected components of $G - V(C)$. Let $N = 8c\left(hd + \binom{h}{2}\right) \cdot \log q$. We claim that there are no N pairwise disjoint members of $\mathcal{F}'$. Suppose for contradiction that $H_1, \ldots, H_N$ are pairwise disjoint members of $\mathcal{F}$. Let A be the minor of G obtained from G by contracting every connected component of $G - V(C)$ having a neighbor in $V(C)$ in to a single vertex, and every H_i into a single vertex. We denote by $u_1, \ldots, u_m$ the vertices resulting the contractions of these connected components of $G - V(C)$, and by $v_1, \ldots, v_N$ the vertices resulting from the contractions of $H_1, \ldots, H_m$. Then, A is a graph with $m + N \leqslant 2N$ vertices, and every v_i for $i \in [N]$ has at least 2^{k-2} neighbors in $u_1, \ldots, u_m$. Hence $|E(A)| \geqslant 2^{k-2}N$. On the other hand, A is a minor of G and so is K_k-minor-free. By Lemma 6.60, this implies that $|E(A)| \leqslant 2^{k-3}|V(A)| - 1 < 2^{k-2}N$. This contradiction proves that there are no N pairwise disjoint members of $\mathcal{F}'$.

Hence, since φ is a (q, c)-good coloring, there exists a set $Z_C \subseteq V(C)$ and a coloring $\psi_C \colon Z_C \to [c \cdot N]$ such that

(i) $V(F) \cap Z_C \neq \emptyset$ for every $F \in \mathcal{F}'$;

(ii) for every connected component C' of $C - Z$, $N_C(V(C'))$ intersects at most two connected components of $C - V(C')$; and

(iii) for every connected subgraph H of C, either $|\varphi(V(H))| > q$, or there is a $(\varphi \times \psi_C)$-center of $V(H) \cap Z$.

Let

$$S = S_0 \cup \bigcup_{C \in \mathcal{C}} Z_C,$$

and for every $u \in S$, let

$$\lambda(u) = \begin{cases} \lambda_0(u) & \text{if } u \in S_0, \\ \psi_C(u) & \text{otherwise, where } C \text{ is the unique member of } \mathcal{C} \text{ containing } u. \end{cases}$$

We now prove (a)-(c). First, $S_0 \subseteq S$ and so $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$ by 6.59.(a). Hence (a) holds.

Let C' be a connected component of $G - S$. Let C be the connected component of $G - S_0$ containing C'. Since C' is disjoint from Z_C, $N_G(V(C'))$ intersects at most $2^{k-2} - 1$ connected components of $G - V(C)$ by (i). Moreover, $N_G(V(C'))$ intersects at most two connected components of $C - V(C')$. Hence $N_G(V(C'))$ intersects at most $2^{k-2}+1 \leqslant 2^{k-1}$ connected components of $G - V(C')$. This proves (b).

Let C be a connected component of $G - S$. Let C' be the connected component of $G - S_0$ containing C'. Let $Y_{0,C'}$ be the set given by 6.59.(c). Let

$$Y_C = Y_{0,C'} \cup [c \cdot N].$$

Note that $|Y_C| \leqslant 56c^2 \left(dh + \binom{h}{2}\right) \cdot \log q$. Let H be a connected subgraph of G intersecting S such that $|\varphi(V(H))| \leqslant q$ and $|\lambda(V(H) \cap S)| \leqslant q$. Suppose that $C_1, \dots, C_m$ are connected components of $G - S$ intersecting $V(H)$. For every $i \in [m]$, we denote by C'_i the connected component of $G - S_0$ containing $V(C_i)$. If H intersects S_0, then by 6.59.(c), there is a $(\varphi \times \lambda_0)$-center u of $V(H) \cap S_0$ with $\lambda_0(u) \in \bigcup_{i\in[m]} Y_{0,C'_i}$, and so u is a $(\varphi \times \lambda)$-center of $V(H) \cap S$ with $\lambda(u) \in \bigcup_{i\in[m]} Y_{C_i}$. Now suppose that H is disjoint from S_0. Then $C'_1 = \cdots = C'_m$. By (iii), there is a $(\varphi \times \psi_{C'_1})$-center u of $V(H) \cap Z_{C'_1}$, and so u is a $(\varphi \times \lambda)$-center of $V(H) \cap S$ with $\lambda(u) \in [c \cdot N] \subseteq Y_{C_i}$ for every $i \in [m]$. This shows (c), and concludes the proof of the lemma. □

We now apply iteratively Lemma 6.61 to obtain a coloring for graphs with no $\mathcal{F}$-rich model of a given apex-forest. We use the same technique as in Claim 6.3.1, while building a suitable coloring. The main inspiration here is Dębski, Felsner, Micek, and Schröder's paper [DMSF21]. See Figure 6.22. The item (c) in Lemma 6.61 will be very useful here, because it basically says that in a connected component C of $G - S$, we can reuse most of the colors, except a few ones which are given in the set Y_C. To formally apply this idea, we maintain a "buffer" of colors $(Y_1, \dots, Y_q)$ which are not yet reusable. Here, Y_a for $a \in [q]$ corresponds to a set of the form Y_C obtained a steps before the current induction call. We will maintain the fact that any connected subgraph between the currently processed connected component and a vertex with color in Y_a must also use colors in each of $Y_1, \dots, Y_{a-1}$, and so already contains at least a colors. When we reach $a = q+1$, we can then actually reuse these colors since we are only interested in connected subgraphs using at most q colors.

Lemma 6.62. *Let c, k be positive integers, let X be a forest. There exists an integer $c_{6.62}(c,k,X)$ such that the following holds. Let q be an integer with $q \geqslant 2$, let G be a K_k-minor-free graph, let φ be a (q,c)-good coloring of G, let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $K_1 \oplus X$. Let $U \subseteq V(G)$, and let C be a connected component of $G - U$. Let Y be a set of at least $2c_{6.62}(c,k,X) \cdot q \log q$ colors. Let $\lambda_0 \colon U \to Y$ and let $Y_1, \dots, Y_q$ be subsets of Y with $|Y_a| \leqslant c_{6.62}(c,k,X) \cdot \log q$ for every $a \in [q]$ such that*

(i) $N_G(V(C))$ *intersects at least one and at most* 2^{k-1} *connected components of* $G - V(C)$*;*

(ii) *for every connected subgraph* H *of* G *intersecting both* U *and* $V(C)$

(1) $|\varphi(V(H))| > q$*, or*

(2) $|\lambda_0(V(H) \cap S)| > q$*, or*

(3) *there exists a* $(\varphi \times \lambda_0)$*-center* u_H *of* $V(H) \cap U$ *and* $a \in [q]$ *such that* $\lambda_0(u_H) \in Y_a$ *and* $|\lambda_0(V(H) \cap U) \cap (\bigcup_{b\in[a]} Y_b)| \geqslant a$*. We fix such a vertex* u_H*.*

Then there exists $S \subseteq V(C) \cup U$ *containing* U *and a coloring* $\lambda\colon S \to Y$ *extending* λ_0 *such that*

(a) $V(F) \cap S \neq \emptyset$ *for every* $F \in \mathcal{F}|_C$*;*

(b) *for every connected component* C' *of* $C - S$*,* $N_G(V(C'))$ *intersects at most* 2^k *connected components of* $G - V(C')$*; and*

(c) *for every connected subgraph* H *of* G *intersecting both* S *and* $V(C)$*,*

(1) $|\varphi(V(H))| > q$*, or*

(2) $|\lambda(V(H) \cap U)| > q$*, or*

(3) *there is a* $(\varphi \times \lambda)$*-center* u *of* $V(H) \cap S$ *such that, if* $V(H) \cap U \neq \emptyset$*, then* $u = u_H$*.*

Proof. By Lemma 6.21, the class of all forests has the coloring elimination property. Therefore, there is a forest X' such that for all sets $S_1, \dots, S_{2^{k-1}} \subseteq V(X')$ whose union is $V(X')$, there exists $i \in [2^{k-1}]$ such that X' contains a S_i-rooted model of $X \sqcup K_1$. Since X' is a forest, there exists positive integers h, d such that $X' \subseteq F_{h,d}$. Let

$$c_{6.62}(c, k, X) = 56c^2 \left(hd + \binom{h}{2}\right).$$

We now proceed by induction on $|V(C)|$. If $\mathcal{F}|_C = \emptyset$, then set $S = U, \lambda = \lambda_0$ and we are done. Now suppose that $\mathcal{F}|_C \neq \emptyset$.

Let $A_1, \dots, A_\ell$ be the connected components of $G - V(C)$ intersecting $N_G(V(C))$. For each $i \in [\ell]$, let $S_i = N_G(V(A_i)) \cap V(C)$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of C such that H contains a member of $\mathcal{F}$ and intersects $\bigcup_{i\in[\ell]} S_i$. We claim that there are no $\mathcal{F}'$-rich model of X' in C. Indeed, if there is such a model, then by the properties of X', there exists $i \in [\ell]$ such that C contains an $\mathcal{F}$-rich model $(B_x \mid x \in V(X \sqcup K_1))$ of $X \sqcup K_1$ such that every branch set has a neighbor in A_i. Then, if x_0 is the vertex of K_1 in $X \sqcup K_1$, then the model $(B_x \mid x \in V(X))$ together with $B_{x_0} \cup A_i$ is an $\mathcal{F}$-rich model of $K_1 \oplus X$ in G, a contradiction. This shows that there is no $\mathcal{F}'$-rich model of X' in C.

Let $\tilde{Y} = Y \setminus \bigcup_{a\in[q]} Y_a$. Note that $|\tilde{Y}| \geqslant 56c^2 \left(hd + \binom{h}{2}\right) \cdot q \log q$. By Lemma 6.61, there exists a set $\tilde{S} \subseteq V(C)$ and a function $\tilde{\lambda}\colon \tilde{S} \to \tilde{Y}$ such that

6.61.(a) $V(F) \cap \tilde{S} \neq \emptyset$ for every $F \in \mathcal{F}'$;

6.61.(b) for every connected component C' of $C - \tilde{S}$, $N_C(V(C'))$ intersects at most 2^{k-1} connected components of $C - V(C')$; and

6.61.(c) there exists a family $\tilde{Y}_{C'} \subseteq \tilde{Y}$ for C' connected component of $C - \tilde{S}$, each of size at most $56c^2 \left(dh + \binom{h}{2}\right) \cdot \log q$, such that for every connected subgraph H of C' intersecting $\tilde{S}$, if $C'_1, \dots, C'_m$ are connected components of $C - \tilde{S}$ intersecting $V(H)$, then

(1) $|\varphi(V(H))| > q$, or
(2) $|\tilde{\lambda}(V(H) \cap \tilde{S})| > q$, or
(3) there is a $(\varphi \times \tilde{\lambda})$-center $\tilde{u}_H$ of $V(H) \cap \tilde{S}$ with $\tilde{\lambda}(\tilde{u}_H) \in \tilde{Y}_{C'_i}$ for every $i \in [m]$. We fix such a vertex $\tilde{u}_H$.

Note that since $\mathcal{F}|_C \neq \emptyset$ and G is connected, $\tilde{S}$ is nonempty. Let

$$U' = U \cup \tilde{S},$$

and let $\lambda'_0 \colon U' \to Y$ be defined by

$$\lambda'_0(u) = \begin{cases} \lambda_0(u) & \text{if } u \in U, \\ \tilde{\lambda}(u) & \text{if } u \in \tilde{S}, \end{cases}$$

for every $u \in U'$. Let H be a connected subgraph of G intersecting U' with $|\varphi(V(H))| \leqslant q$ and $|\lambda'_0(V(H) \cap U')| \leqslant q$. Note that in particular $|\lambda_0(V(H) \cap U)| \leqslant q$ and $|\tilde{\lambda}(V(H) \cap \tilde{U}| \leqslant q$. If $V(H) \cap U \neq \emptyset$, then let $u'_H = u_H$, and if $V(H) \cap U = \emptyset$, then let $u'_H = \tilde{u}_H$.

Let $\mathcal{C}$ be the family of all the connected components of $C - \tilde{S}$ that contain a member of $\mathcal{F}$. Let $C' \in \mathcal{C}$. Let $\tilde{Y}_{C'}$ be the set given by 6.61.(c). Let $(Y'_1, \dots, Y'_q) = (\tilde{Y}_{C'}, Y_1, \dots, Y_{q-1})$. We now show that $U', C', \lambda'_0, Y, (Y'_1, \dots, Y'_q), (u'_H)_H$, satisfies the hypothesis of the lemma, and so we can call induction. First observe that, since $C' \notin \mathcal{F}'$, we have $N_G(V(C')) \cap U = \emptyset$, and so $N_G(V(C'))$ intersects at most 2^{k-1} connected components of $G - V(C')$ by 6.61.(b). Let H be a connected subgraph of G intersecting both U' and $V(C')$, and such that $|\varphi(V(H))| \leqslant q$ and $|\lambda'_0(V(H) \cap U')| \leqslant q$. Let $C'_1, \dots, C'_m$ be connected components of $C - \tilde{S}$ intersecting $V(H)$ with $C'_1 = C'$. First suppose that $V(H)$ intersects U. By hypothesis, $u'_H = u_H$ is a $(\varphi \times \lambda_0)$-center of $V(H) \cap U$ such that, for some $a \in [q]$, we have $\lambda_0(u'_H) \in Y_a$. Since the colors used by λ_0 and $\tilde{\lambda}$ are pairwise distinct, u'_H is a $(\varphi \times \lambda'_0)$-center of $V(H) \cap U'$. Moreover, since C is a member of $\mathcal{F}$, the set $V(H)$ intersects $\tilde{S}$. By hypothesis, $|\lambda_0(V(H) \cap U) \cap (\bigcup_{b \in [q]} Y_b)| \geqslant a$. Since $\tilde{Y}$ is disjoint from $\bigcup_{b \in [a]} Y_b$, we deduce that

$$\begin{aligned} |\lambda'_0(V(H) \cap U') \cap (\textstyle\bigcup_{b \in [a+1]} Y'_b)| &= |\lambda_0(V(H) \cap U) \cap (\textstyle\bigcup_{b \in [a]} Y_b)| + |\lambda'_0(V(H) \cap U') \cap \tilde{Y}_{C'}| \\ &\geqslant a + 1. \end{aligned}$$

If $a = q$, then we have $|\lambda'_0(V(H) \cap U')| > q$, a contradiction. Otherwise, since $\lambda'_0(u) \in Y_a = Y'_{a+1}$, the vertex u'_H is as wanted. Now assume that $V(H)$ is disjoint from U. Then $V(H)$ is a connected subgraph of C. Hence, by 6.61.(c), and because $|\varphi(V(H))| \leqslant q$ and $|\tilde{\lambda}(V(H) \cap \tilde{S})| \leqslant q$, the vertex u'_H is a $(\varphi \times \tilde{\lambda})$-center of $V(H) \cap \tilde{S} = V(H) \cap U'$ such that $\lambda'_0(u'_H) \in \tilde{Y}_{C'_i}$ for every $i \in [m]$. Since $\lambda'_0(u'_H) \in Y'_1$, we clearly have $|\lambda'_0(V(H) \cap U) \cap (\bigcup_{b \in [1]} Y'_b)| \geqslant 1$. This proves that $U', C', Y, \lambda'_0, (Y'_1, \dots, Y'_q), (u'_H)_H$ satisfies the hypothesis of the lemma.

Since $\mathcal{F}' \neq \emptyset$, the set $\tilde{S}$ is nonempty, and so $|V(C')| < |V(C)|$. Hence we can call the induction hypothesis on $U', C', Y, \lambda'_0, (Y'_1, \dots, Y'_q)$. This gives a set $S_{C'} \subseteq V(C') \cup U'$ containing U' and a coloring $\lambda_{C'} \colon S_{C'} \to Y$ extending λ'_0 such that

(a') $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}|_{C'}$;

(b') for every connected component C'' of $C' - S_{C'}$, $N_G(V(C''))$ intersects at most 2^k connected components of $G - V(C'')$; and (c') for every connected subgraph H of G intersecting $V(C')$ and $S_{C'}$, either

(1) $|\varphi(V(H))| > q$, or

(2) $|\lambda_{C'}(V(H) \cap S)| > q$, or

(3) there is a $(\varphi \times \lambda_{C'})$-center u of $V(H) \cap S_{C'}$ such that, if $V(H) \cap U' \neq \emptyset$, then $u = u'_H$.

Then let

$$S = \bigcup_{C' \in \mathcal{C}} S_{C'}$$

and let $\lambda \colon S \to Y$ be defined by

$$\lambda(u) = \lambda_{C'}(u)$$

for every $u \in S_{C'}$, for every $C' \in \mathcal{C}$. This is well defined because the colorings $\lambda_{C'}$ for $C' \in \mathcal{C}$ coincide on U'. We now show (a)-(c).

Let $F \in \mathcal{F}$ intersecting C. If $V(F) \cap U' \neq \emptyset$ then $V(F) \cap S \neq \emptyset$. Otherwise, there exists $C' \in \mathcal{C}$ such that $F \subseteq C'$. Then, by (a'), $V(F) \cap S_{C'} \neq \emptyset$ and so $V(F) \cap S \neq \emptyset$. This proves (a).

Let C'' be a connected component of $C - S$. Since C'' is disjoint from $\tilde{S}$, either $N_G(V(C'')) \subseteq U \cup \tilde{S}$, or there exists $C' \in \mathcal{C}$ such that $C'' \subseteq C'$. In the first case, $N_G(V(C''))$ intersects at most 2^{k-1} connected components of $G - V(C)$ by assumption, and at most 2^{k-1} connected components of $C - \tilde{S}$ by 6.61.(b). This implies that $N_G(V(C''))$ intersects at most 2^k connected components of $G - V(C'')$. In the second case, let $C' \in \mathcal{C}$ such that $C'' \subseteq C'$. Then, by (b'), $N_G(V(C''))$ intersects at most 2^k connected components of $G - V(C'')$. This proves (b).

Let H be a connected subgraph of G intersecting both $V(C)$ and S such that $|\varphi(V(H))| \leqslant q$ and $|\lambda(V(H) \cap S)| \leqslant q$. Let $C'_1, \dots, C'_m$ be the connected components of $C - \tilde{S}$ intersecting $V(H)$ (with possibly $m = 0$). First suppose that $V(H)$ intersects U. By hypothesis, the vertex $u'_H = u_H$ is a $(\varphi \times \lambda_0)$-center of $V(H) \cap S_{C'_i}$ for every $i \in [m]$. Therefore, u_H is a $(\varphi \times \lambda)$-center of $V(H) \cap S$. Now suppose that $V(H)$ is disjoint from U. Assume that $V(H)$ intersects $\tilde{S}$. By (c'), the vertex u'_H is a $(\varphi \times \tilde{\lambda})$-center u of $V(H) \cap S_{C'_i}$ for every $i \in [m]$, and so u'_H is a $(\varphi \times \lambda)$-center of $V(H) \cap S$. Finally, assume that $V(H)$ is disjoint from $\tilde{S} \cup U$. Then, there exists $C' \in \mathcal{C}$ such that $H \subseteq C'$. Therefore, by (c'), there is a $(\varphi \times \lambda_{C'})$-center u of $V(H) \cap S_{C'}$. By the definition of λ, and because $V(H) \cap S_{C'} = V(H) \cap S$, we conclude that u is a $(\varphi \times \lambda)$-center of $V(H) \cap S$. This shows (c) and concludes the proof of the lemma. □

We can now deduce the main result of this section.

Lemma 6.63. *Let k, c be positive integer. There exists a constant α such that for every $X \in \mathbf{A}(\mathcal{R}_2)$, there exists a constant $\beta(X)$ such that the following holds. For every K_k-minor-free graph G, for every integer q with $q \geqslant 2$, for every (q, c)-good coloring φ of G, for every family $\mathcal{F}$ of connected subgraphs of G, for every integer q with $q \geqslant 2$, if there is no $\mathcal{F}$-rich model of X in G, then there exists $S \subseteq V(G)$ such that*

(a) *$V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$,*

(b) *for every connected component C of $G - S$, $N_G(V(C))$ intersects at most α connected components of $G - V(C)$, and*

(c) $\mathrm{cen}_q(G, \varphi, S) \leqslant \beta(X) \cdot q \log q$.

Proof. Let $\alpha = 2^k$, and let $X \in \mathbf{A}(\mathcal{R}_2)$. There exists $X' \in \mathcal{R}_2$ such that $X \subseteq K_1 \oplus X'$. Fix such a forest X', and let $\beta(X) = c_{6.62}(c,k,X')$. Let q be an integer with $q \geqslant$, let G be a K_k-minor-free graph, let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of X in G.

Let Y be a set of $c_{6.62}(c,k,X') \cdot q \log q$ colors, let $y_0 \in Y$ arbitrary, let $Y_1 = \{y_0\}$ and $Y_2 = \cdots = Y_q = \emptyset$.

Let $\mathcal{C}$ be the family of all the connected components of G. Let $C \in \mathcal{C}$, let U_C be an arbitrary singleton in $V(C)$, and let $\lambda_0 \colon U \to \{y_0\}$. For every connected subgraph H of C intersecting U_C, let u_H be the unique vertex in U_C.

Let $\mathcal{C}_C$ be the family of all the connected components of $C - U_C$, and let $C' \in \mathcal{C}_C$. By Lemma 6.62 there exists $S_{C'} \subseteq V(C') \cup U_C$ containing U_C and a coloring $\lambda_{C'} \colon S_{C'} \to Y$ extending λ_0 such that

6.62.(a) $V(F) \cap S_{C'} \neq \emptyset$ for every $F \in \mathcal{F}|_{C'}$;

6.62.(b) for every connected component C'' of $C' - S_{C'}$, $N_C(V(C''))$ intersects at most 2^k connected components of $C - V(C'')$; and

6.62.(c) for every connected subgraph H of C intersecting $V(C'')$ and $S_{C'}$, either

(1) $|\varphi(V(H))| > q$, or
(2) $|\lambda_{C'}(V(H) \cap S)| > q$, or
(3) there is a $(\varphi \times \lambda_{C'})$-center u of $V(H) \cap S$, and if $V(H) \cap U \neq \emptyset$, then $u = u_H$.

Then the coloring $\lambda_C \colon \bigcup_{C' \in \mathcal{C}_C} S_{C'} \to Y$ defined by $\lambda(u) = \lambda_{C'}(u)$ for every $C' \in \mathcal{C}_C$ and $u \in S_{C'}$, witnesses the fact that $\operatorname{cen}_q(C, \varphi|_{V(C)}, \bigcup_{C' \in \mathcal{C}_C} S_{C'}) \leqslant c_{6.62}(c,k,X') \cdot q \log q$.

Then, by Observation 6.11, we conclude that the set

$$S = \bigcup_{C \in \mathcal{C}} \bigcup_{C' \in \mathcal{C}_C} S_{C'}$$

satisfies

$$\operatorname{cen}_q(G, \varphi, S) \leqslant c_{6.62}(c,k,X') \cdot q \log q = \beta(X) \cdot q \log q,$$

which proves (c).

Moreover, by 6.62.(a), $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and for every connected components C of $G - S$, $N_G(V(C))$ intersects at most $2^k = \alpha$ connected components of $G - V(C)$ by 6.62.(b). This proves (a) and (b). □

6.9.5 Applying the induction

We can now prove Theorem 1.34, which is covered by the following two theorems. See Figures 6.2 and 6.3 for a general view of these proofs.

Theorem 6.64. *Let t be an integer with $t \geqslant 2$, and let $X \in \mathcal{S}_t$. There is constant a such that, for every X-minor-free graph G, for every positive integer q,*

$$\operatorname{cen}_q(G) \leqslant a \cdot (\operatorname{tw}(G) + 1) \cdot q^{t-2},$$

and

$$\operatorname{cen}_q(G) \leqslant a \cdot q^{t-1}.$$

Proof. Let $k = |V(X)|$. By Lemma 6.56, there exists a positive integer c such that for every positive integer q, every K_k-minor-free graph admit a (q,c)-good coloring using $q+1$ colors. We assume $c \geqslant 2$. For every positive integer q, for every graph G, for every $S \subseteq V(G)$, let

$$\mathrm{par}_q(G,S) = \max_{\varphi\ (q,c)\text{-good coloring of } G} \mathrm{cen}_q(G,\varphi,S).$$

We show by induction that for every integer t with $t \geqslant 2$, $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding. By Lemma 6.14, $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ is a nice family of focused parameters. We first show the cases $t = 2$ and $t = 3$. By the definition of (q,c)-good colorings, $q \mapsto 1$ is $(\mathrm{par}, \mathcal{R}_1)$-bounding. Then, Theorem 6.23 implies that $q \mapsto 1$ is $(\mathrm{par}, \mathcal{S}_2)$-bounding. By Theorem 6.3, because $\mathcal{S}_2$ is closed under disjoint union and has the coloring elimination property by Lemma 6.21, we deduce that the function $q \mapsto q$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-bounding. By Theorem 6.24, the function $q \mapsto q$ is $(\mathrm{par}, \mathcal{S}_3)$-bounding.

Now suppose $t \geqslant 4$. By the induction hypothesis, $q \mapsto q^{t-3}$ is $(\mathrm{par}, \mathcal{S}_{t-1})$-bounding. Then, by Theorem 6.3, because $\mathcal{S}_{t-1}$ is closed under disjoint union and has the coloring elimination property by Lemma 6.21, $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_{t-1}))$-bounding. By Theorem 6.4, because $\mathcal{S}_{t-1}$ is closed under disjoint union, this implies that $q \mapsto q^{t-2}$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding. This concludes the proof of the induction.

By the definition of $(\mathrm{par}, \mathcal{S}_t)$-bounding functions applied for $k = |V(X)|$, there exists a positive integer β such that the following holds. Let G be an X-minor-free graph, and let $\mathcal{F}$ be the family of all the one-vertex subgraphs of G. Note there is no $\mathcal{F}$-rich model of X in G. For every positive integer q, there exists $S \subseteq V(G)$ such that (in particular)

(i) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so $S = V(G)$, and

(ii) $\mathrm{par}_q(G,S) \leqslant \beta \cdot q^{t-2}$.

We deduce that

$$\max_{\varphi\ (q,c)\text{-good coloring of } G} \mathrm{cen}_q(G,\varphi,V(G)) \leqslant \beta \cdot q^{t-2}.$$

Now, by Lemma 6.55, there is a (q,c)-good coloring φ of G using $\mathrm{tw}(G)+1$ colors. Let $\psi \colon V(G) \to [\beta \cdot q^{t-2}]$ witnessing the fact that $\mathrm{cen}_q(G,\varphi,V(G)) \leqslant \beta \cdot q^{t-2}$. We claim that $\varphi \times \psi$ is a q-centered coloring of G. Indeed, for every $H \subseteq G$ connected, by the definition of ψ, one of the following holds

(i) $|\varphi(V(H))| > q$, and so $|(\varphi \times \psi)(V(H))| > q$,

(ii) $|(\varphi \times \psi)(V(H))| > q$, or

(iii) there is a $(\varphi \times \psi)$-center of $V(H)$.

Therefore, $\varphi \times \psi$ is a q-centered coloring using at most $\beta \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2}$ colors. This proves that

$$\mathrm{cen}_q(G) \leqslant \beta \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2}.$$

By the definition of c, there is a (q,c)-good coloring of G using $q+1$ colors. Let $\psi \colon V(G) \to [\beta \cdot q^{t-2}]$ witnessing the fact that $\mathrm{cen}_q(G,\varphi,V(G)) \leqslant \beta \cdot q^{t-2}$. For the same reason as above, $\varphi \times \psi$ is a q-centered coloring of G, and we conclude that

$$\mathrm{cen}_q(G) \leqslant \beta \cdot (q+1) \cdot q^{t-2} \leqslant 2\beta \cdot q^{t-1}.$$

This proves the theorem for $a = 2\beta$. □

Theorem 6.65. *Let t be an integer with $t \geqslant 3$, and let $X \in \mathcal{R}_t$. There is constant a such that, for every X-minor-free graph G, for every integer q with $q \geqslant 2$,*

$$\mathrm{cen}_q(G) \leqslant a \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q,$$

and

$$\mathrm{cen}_q(G) \leqslant a \cdot q^{t-1} \log q.$$

Proof. Let $k = |V(X)|$. By Lemma 6.56, there exists a positive integer c such that for every positive integer q, every K_k-minor-free graph admit a (q,c)-good coloring using $q+1$ colors. We assume $c \geqslant 2$. For every positive integer q, for every graph G, for every $S \subseteq V(G)$, let

$$\mathrm{par}_q(G,S) = \max_{\varphi\ (q,c)\text{-good coloring of } G} \mathrm{cen}_q(G,\varphi,S).$$

We show by induction that for every integer t with $t \geqslant 3$, $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}, \mathcal{S}_t)$-bounding. By Lemma 6.14, $(\mathrm{par}_q \mid q \in \mathbb{N}_{>0})$ is a nice family of focused parameters. When $t = 3$, by Lemma 6.63, the function $q \mapsto q\log(q+1)$ is $(\mathrm{par}, \mathbf{A}(\mathcal{R}_2))$-bounding. If $t \geqslant 4$ and $q \mapsto q^{t-3}\log(q+1)$ is $(\mathrm{par}, \mathcal{R}_{t-1})$-bounding, then, by Theorem 6.4, and because $\mathcal{R}_{t-1}$ is closed under disjoint union, the function $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}, \mathcal{R}_t)$-bounding. This concludes the proof of the induction.

By the definition of $(\mathrm{par}, \mathcal{R}_t)$-bounding functions applied for $k = |V(X)|$, there exists a positive integer β such that the following holds. Let G be an X-minor-free graph, and let $\mathcal{F}$ be the family of all the one-vertex subgraphs of G. Note there is no $\mathcal{F}$-rich model of X in G. For every positive integer q, there exists $S \subseteq V(G)$ such that (in particular)

(i) $V(F) \cap S \neq \emptyset$ for every $F \in \mathcal{F}$, and so $S = V(G)$, and

(ii) $\mathrm{par}_q(G,S) \leqslant \beta \cdot q^{t-2}\log(q+1)$.

We deduce that

$$\max_{\varphi\ (q,c)\text{-good coloring of } G} \mathrm{cen}_q(G,\varphi,V(G)) \leqslant \beta \cdot q^{t-2}\log(q+1).$$

Now, by Lemma 6.55, there is a (q,c)-good coloring φ of G using $\mathrm{tw}(G)+1$ colors. Let $\psi \colon V(G) \to [\lfloor \beta \cdot q^{t-2}\log(q+1)\rfloor]$ witnessing the fact that $\mathrm{cen}_q(G,\varphi,V(G)) \leqslant \beta \cdot q^{t-2}\log(q+1)$. We claim that $\varphi \times \psi$ is a q-centered coloring of G. Indeed, for every $H \subseteq G$ connected, by the definition of ψ, one of the following holds

(i) $|\varphi(V(H))| > q$, and so $|(\varphi\times\psi)(V(H))| > q$, or

(ii) $|(\varphi\times\psi)(V(H))| > q$, or

(iii) there is a $(\varphi\times\psi)$-center of $V(H)$.

Therefore, $\varphi\times\psi$ is a q-centered coloring using at most $\beta\cdot(\mathrm{tw}(G)+1)\cdot q^{t-2}$ colors. This proves that

$$\mathrm{cen}_q(G) \leqslant \beta \cdot (\mathrm{tw}(G)+1)\cdot q^{t-2}\log(q+1).$$

By the definition of c, there is a (q, c)-good coloring of G using $q + 1$ colors. Let $\psi\colon V(G) \to [\lfloor \beta \cdot q^{t-2} \log(q + 1)\rfloor]$ witnessing the fact that $\mathrm{cen}_q(G, \varphi, V(G)) \leqslant \beta \cdot q^{t-2} \log(q + 1)$. For the same reason as above, $\varphi \times \psi$ is a q-centered coloring of G, and we conclude that

$$\mathrm{cen}_q(G) \leqslant \beta \cdot (q + 1) \cdot q^{t-2} \log(q + 1) \leqslant 2\beta \cdot q^{t-1} \log(q + 1).$$

When $q \geqslant 2$, we have $\log(q + 1) \leqslant 2 \log(q)$, and so the theorem holds for $a = 4\beta$. $\square$

PART III

Conclusion

CHAPTER 7

Conclusion and open problems

In this thesis, I studied several questions of the form: which structural property on graphs $X_1, \dots, X_\ell$ ensures that $\{X_1, \dots, X_\ell\}$-minor-free graphs have a given structure. The structures on $\{X_1, \dots, X_\ell\}$-minor-free graphs I considered were: bounded k-treedepth in Chapter 2, bounded layered pathwidth and bounded layered treedepth in Chapter 4, a product structure in Chapter 5, and polynomially bounded centered chromatic numbers and weak coloring numbers in Chapter 6. To conclude, I propose as open problems several generalizations and improvements of these results.

7.1 Minor-monotone graph parameters

One of the motivations behind the systematic study of obstructions for minor-monotone graph parameters is the following conjecture by Robertson, Seymour, and Thomas [RST96].

Conjecture 7.1 (Robertson, Seymour, and Thomas [RST96])**.** *For every minor-monotone graph parameter* p*, there exist finitely many minor-closed classes of graphs* $\mathcal{X}_1, \dots, \mathcal{X}_\ell$ *such that for every minor-closed class of graphs* $\mathcal{C}$*, the following are equivalent:*

(1) p *is bounded in* $\mathcal{C}$*, that is, there exists an integer* N *such that* $\mathrm{p}(G) \leqslant N$ *for every* $G \in \mathcal{C}$*,*

(2) $\mathcal{C}$ *excludes a member of* $\mathcal{X}_i$*, that is* $\mathcal{X}_i \not\subseteq \mathcal{C}$*, for every* $i \in [\ell]$*.*

For example, for $\mathrm{p} = \mathrm{tw}$, one can take $\ell = 1$, and for $\mathcal{X}_1$ the class of planar graphs, since a class of graphs has bounded treewidth if and only if it excludes a planar graph (Grid-Minor Theorem). Conjecture 7.1 is a very special case of a broader conjecture asserting that infinite countable graphs form a well-quasi-ordering for the graph minor relation. This should be seen as an extension for graph parameters of Robertson-Seymour Theorem, which asserts that every minor-closed class of graphs can be characterized by finitely many excluded minors. See the recent article [PPT23a] by Paul, Protopapas, and Thilikos for more information on this topic, and some possible algorithmic applications.

The importance of Conjecture 7.1 motivates a more systematic study of minor-monotone parameters. In Chapter 2, we characterized classes of graphs having bounded k-treedepth in terms of excluded minors. It would be interesting to obtain such results for broader families of minor-monotone parameters. Recall that, informally, k-treedepth measures how a graph can be built by alternatively adding a vertex and performing $(< k)$-clique-sums. In the remaining of this section, we define a large family of minor-monotone graph parameters using these two operations. These parameters will be indexed by the languages over an alphabet Σ which describes which sequences of vertex additions and clique-sums are allowed to construct the input graph.

Let Σ be the alphabet containing the letters a and s_k for all $k \in \mathbb{N}_{>0}$. We denote by ε the empty word in Σ^*. Intuitively, the letter a corresponds to a vertex addition, and the letter s_k to $(< k)$-clique-sums. We will identify regular expressions over Σ with their corresponding languages. For every $w \in \Sigma^*$, we define p_w as the graph parameter taking values in $\mathbb{N}_{\geqslant 0} \cup \{+\infty\}$ defined inductively as follows. For every graph G, for every $w \in \Sigma^*$, for every positive integer k,

$$\begin{aligned}
\mathrm{p}_\varepsilon(G) &= \begin{cases} 0 & \text{if } G = \emptyset, \\ +\infty & \text{otherwise,} \end{cases} \\
\mathrm{p}_{wa}(G) &= \min\left(\{\mathrm{p}_w(G)\} \cup \{\mathrm{p}_w(G-u)+1 \mid u \in V(G)\}\right), \\
\mathrm{p}_{ws_k}(G) &= \min\left(\{\mathrm{p}_w(G)\} \cup \{\max\{\mathrm{p}_{ws_k}(G_1), \mathrm{p}_{ws_k}(G_2)\} \mid (G_1,G_2)\}\right)
\end{aligned}$$

where in the last equality, (G_1, G_2) ranges over all the pairs of graphs such that G is a $(< k)$-clique-sum of G_1 and G_2 with $|V(G_1)|, |V(G_2)| > |V(G_1) \cap V(G_2)|$. Less formally, we read from left to right w as follows. When reading the letter a, we can add a vertex (with arbitrary neighborhood) to a previously obtained graph, and when reading the letter s_k, we can make arbitrarily many $(< k)$-clique-sums of previously obtained graphs. Then $\mathrm{p}_w(G)$ is the minimum number of times we add a vertex (when reading the letter a) in such a construction of G. If it is not possible to construct G this way, then $\mathrm{p}_w(G) = +\infty$. Now, for every $L \subseteq \Sigma^*$, let

$$\mathrm{p}_L(G) = \min_{w \in L} \mathrm{p}_w(G),$$

with the convention $\min \emptyset = +\infty$. An induction as in the proof of Lemma 2.5 shows that for every $L \subseteq \Sigma^*$, the parameter p_L is minor-monotone. Note that $\mathrm{p}_L(G) \geqslant \mathrm{tw}(G) + 1$ for every graph G, for every $L \subseteq \Sigma^*$. We denote by $s_{+\infty}$ the language $\{s_k \mid k \geqslant 1\}$. Several classical minor-monotone graph parameters are of the form p_L for $L \subseteq \Sigma^*$: for every graph G with at least one edge, for every positive integer k,

$$\begin{aligned}
\mathrm{p}_{a^*}(G) &= |V(G)|, \\
\mathrm{p}_{a^*s_1}(G) &= \max_{\substack{C \text{ connected} \\ \text{component of } G}} |V(C)|, \\
\mathrm{p}_{a^*s_2}(G) &= \max_{B \text{ block of } G} |V(B)|, \\
\mathrm{p}_{a^*s_k}(G) &= \mathrm{tw}_k(G) + 1,
\end{aligned}$$

where $\mathrm{tw}_k(G)$ is the k-treewidth of G, which is defined in [GJ16] as the minimum width of a tree decomposition of G of adhesion less than k,

$$\begin{aligned}
\mathrm{p}_{a^*s_{+\infty}}(G) &= \mathrm{tw}(G)+1,\\
\mathrm{p}_{(as_k)^*}(G) &= \mathrm{td}_k(G),\\
\mathrm{p}_{(as_1)^*s_2}(G) &= \max_{B \text{ block of } G} \mathrm{td}(B),\\
\mathrm{p}_{as_1a^*}(G) &= \mathrm{vc}(G)+1, && \text{where } \mathrm{vc}(G) \text{ is the vertex cover number of } G,\\
\mathrm{p}_{a^2s_2a^*}(G) &= \mathrm{fvs}(G)+2, && \text{where } \mathrm{fvs}(G) \text{ is the feedback vertex set number of } G.
\end{aligned}$$

The last two examples follows from the following property: for every graph G,

$$\mathrm{p}_{as_1}(G) = \begin{cases} 0 & \text{if } G=\emptyset,\\ 1 & \text{if } E(G)=\emptyset \text{ and } V(G)\neq\emptyset,\\ +\infty & \text{if } E(G)\neq\emptyset, \end{cases}$$

and

$$\mathrm{p}_{a^2s_2}(G) = \begin{cases} 0 & \text{if } G=\emptyset,\\ 1 & \text{if } E(G)=\emptyset \text{ and } V(G)\neq\emptyset,\\ 2 & \text{if } E(G)\neq\emptyset \text{ and } G \text{ is a forest},\\ +\infty & \text{otherwise.} \end{cases}$$

Another noteworthy example is $\mathrm{p}_{a^2s_2(as_1)^*}$, which is (two plus) the elimination distance to a forest, a parameter used in the context of parameterized complexity, see [DJ24].

Problem 7.2. *Given a language $L \subseteq \Sigma^*$, characterize classes of graphs having bounded p_L in terms of excluded minors.*

There is little hope that a general answer can be given to this problem. Indeed, for every nonnegative integer k, for every graph G,

$$\mathrm{p}_{a^{k+1}s_{+\infty}}(G) = \begin{cases} \mathrm{tw}(G)+1 & \text{if } \mathrm{tw}(G)\leqslant k,\\ +\infty & \text{otherwise,} \end{cases}$$

and so a class of graphs has bounded $\mathrm{p}_{a^{k+1}s_{+\infty}}$ if and only if it excludes all the minimal obstructions to treewidth at most k, which have not been determined yet, even for $k=4$ (see [Ram97]). Nonetheless, proving Conjecture 7.1, that is just the existence of such a characterization, for the parameters p_L for $L \subseteq \Sigma^*$ would be very interesting.

An important observation is that it is enough to consider only very specific languages $L \subseteq \Sigma^*$, namely the downward closed languages under a specific order $\preccurlyeq^*$ on Σ^*, that we now define. Let $\preccurlyeq$ be the order on Σ whose relations are $a \preccurlyeq a$ and $s_i \preccurlyeq s_j$ for every positive integers i,j with $i \leqslant j$, and let $\preccurlyeq^*$ be the order on Σ^* defined by, for all $x_1,\dots,x_\ell,y_1,\dots,y_m \in \Sigma$, $x_1\dots x_\ell \preccurlyeq^* y_1\dots y_m$ if and only if there exist $1 \leqslant i_1 < \cdots < i_\ell \leqslant m$ such that $x_j \preccurlyeq y_{i_j}$ for every $j \in [\ell]$. It follows from the definition that $\mathrm{p}_w(G) \geqslant \mathrm{p}_{w'}(G)$ for every graph G and for every $w,w' \in \Sigma^*$ with $w \preccurlyeq^* w'$. Hence, for every $L \subseteq \Sigma^*$, $\mathrm{p}_L = \mathrm{p}_{\downarrow L}$, where $\downarrow L$ is the language $\{u \in \Sigma^* \mid \exists w \in L, u \preccurlyeq^* w\}$. Moreover, by Higman's Lemma [Hig52], $(\Sigma^*, \preccurlyeq^*)$ is a well-quasi-ordering. Hence for every $L \subseteq \Sigma^*$, there exists a finite set $F \subseteq \Sigma^*$ (namely the set of all the minimal elements of $\Sigma^* \setminus \downarrow L$) such that $\downarrow L = \{w \in \Sigma^* \mid \forall u \in F, u \not\preccurlyeq^* w\}$. In particular, p_L is computable for every fixed $L \subseteq \Sigma^*$.

Another direction of research is to obtain good bounds for the k-treedepth of a graph G as a function of the maximum size of a minor of G of the form $T \square P_\ell$ for T a tree on k vertices. We proved in Chapter 2 that the former is bounded by a function of the latter, but this function is very large, and we did not compute it explicitly. We conjecture that a bound polynomial in both k and ℓ holds.

Conjecture 7.3. *There exists a polynomial function $f\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ such that, for every graph G, for all positive integer k, ℓ, either*

(i) *there is a tree T on k vertices such that $T \square P_\ell$ is a minor of G, or*

(ii) $\mathrm{td}_k(G) \leqslant f(\ell + k)$.

Using the same technique as in Section 2.8, one can show that this would imply the Polynomial Grid-Minor Theorem [CC16, CT21]: for every positive integer ℓ, graphs that do not contain the $\ell \times \ell$ as a minor have treewidth bounded from above by a polynomial function of ℓ.

7.2 Rooted minors

In Chapter 4, we proved several structural properties for graphs excluding a specific graph as a rooted minor. The general problem is to describe, given a graph X, the structure of a graph G "focused" on a subset S of vertices if G has no S-rooted model of X. We investigated the cases where X is a path (Theorem 4.2), and a tree (Theorem 4.5). The obtained bounds do not precisely match the lower bounds, and we leave as open problems to bridge these gaps.

Within Theorem 4.2, we show that for every positive integer ℓ, for every graph G, and for every $S \subseteq V(G)$, if G has no S-rooted model of P_ℓ, then $\mathrm{td}(G, S) \leqslant \binom{\ell}{2}$. A first question is whether this can be improved to $\ell - 1$.

Conjecture 7.4. *For every positive integer ℓ, for every graph G, for every $S \subseteq V(G)$, if there is no S-rooted model of P_ℓ in G, then*

$$\overline{\mathrm{td}}(G, S) \leqslant \ell - 1.$$

This would be tight as shown by $G = K_{\ell-1}$ and $S = V(G)$.

Within Theorem 4.5, we show that for every forest F with at least one vertex, for every graph G, for every $S \subseteq V(G)$, if G has no S-rooted model of F, then $\overline{\mathrm{pw}}(G, S) \leqslant 2|V(F)| - 2$. A second problem is whether this factor two can be removed.

Conjecture 7.5. *For every nonnull forest F, for every graph G, for every $S \subseteq V(G)$, if there is no S-rooted model of F in G, then*

$$\overline{\mathrm{pw}}(G, S) \leqslant |V(F)| - 2.$$

Again, this would be tight as shown by $G = K_{|V(F)|-1}$ and $S = V(G)$.

Within Corollary 4.17, we show the Erdős-Pósa property for S-rooted trees with a bound $(2k|V(T)| - 1)(k - 1) = \mathcal{O}(k^2)|V(T)|$. I tend to believe that the bound $|V(T)| \cdot (k - 1)$ holds.

Conjecture 7.6. *For every tree T, for every positive integer k, for every graph G, for every $S \subseteq V(G)$, if there is no S-rooted model of $k \cdot T$ in G, then there exists $X \subseteq V(G)$ intersecting every S-rooted model of T in G with*

$$|X| \leqslant |V(T)| \cdot (k-1).$$

This would be tight as shown by $G = K_{|V(T)|k-1}$ and $S = V(G)$. For $T = K_2$, this is a celebrated theorem of Gallai about the so-called "S-paths" [Gal64]. Another known interesting case is when $S = V(G)$, which was proved by Dujmović, Joret, Micek, and Morin in [DJMM24].

Overall, I think that the notion of rooted minors is interesting in itself, and so deserves a more systematic study. A noteworthy statement in this direction is the following, which informally says that, for all graph G and $S \subseteq V(G)$, if G excludes an S-rooted minor, then there is a torso of a superset of S that excludes a slightly larger minor. Here, we use the notation $\mathrm{torso}_G(W)$ for a set $W \subseteq V(G)$ to denote the graph with vertex set W and where two vertices u, v are adjacent if and only if there is a path from u to v in G internally disjoint from W.[1]

Theorem 7.7 ([DHH$^+$24, Lemma 14])**.** *Let t be a positive integer, let G be a graph, and let $S \subseteq V(G)$. If there is no S-rooted model of K_t in G, then there exists $S' \subseteq V(G)$ with $S \subseteq S'$ such that $\mathrm{torso}_G(S')$ is K_{2t-1}-minor-free.*

The statement of Lemma 14 in [DHH$^+$24] is actually slightly different, but its proof shows Theorem 7.7. For sake of completeness, we provide a proof in Appendix B.

In light of these results, it is tempting to believe that, for every finite family $\mathcal{X}$ of graphs,

for every graph G, and for every $S \subseteq V(G)$, if for every $X \in \mathcal{X}$, G has no S-rooted model of X, then there exists $S' \subseteq V(G)$ containing S such that $\mathrm{torso}_G(S')$ is $\mathcal{X}$-minor-free. (7.1)

Such a property would imply Conjectures 7.4 to 7.6 by reducing the problem to the non-rooted case. However, this does no holds for every $\mathcal{X}$. To see that, consider $\mathcal{X} = \{K_4\}$. Let G be the $\ell \times \ell$ grid for some positive integer ℓ with $\ell \geqslant 4$, and let S be the vertex set of the outer face of G (i.e. the unique face of G of size $4(\ell-1)$). Then, since K_4 is not outer-planar, G has no S-rooted model of K_4. However, $\overline{\mathrm{tw}}(G,S) \geqslant \ell - 1$ (see Section 4.4). Since K_4-minor-free graphs have treewidth at most 2, and because $\overline{\mathrm{tw}}(G,S) = \min_{S' \subseteq V(G), S \subseteq S'} \mathrm{tw}(\mathrm{torso}_G(S))$, we deduce that there is no $S' \subseteq V(G)$ containing S with $\mathrm{torso}_G(S')$ K_4-minor-free. We leave as an open problem to determine for which family $\mathcal{X}$ the statement (7.1) holds.

Problem 7.8. *Characterize the finite families of graphs $\mathcal{X}$ for which* (7.1) *holds.*

7.3 Weak coloring numbers and centered chromatic numbers

In Chapter 6, we proved generic bounds on the weak coloring numbers and centered chromatic numbers of minor-closed classes of graphs which are tight up to a $\mathcal{O}(q)$ factor. Closing this $\mathcal{O}(q)$ gap is a very challenging open problem.

[1] Note that $\mathrm{torso}_G(W) = \mathrm{torso}_{G,\mathcal{W}}(W) = \mathrm{torso}_G(W, \mathcal{B})$ where $\mathcal{B} = \{V(C) \cup N_G(V(C)) \mid C$ component of $G - W\}$, and $\mathcal{W}$ is the tree decomposition of G indexed by a star whose central bag is W and the other bags are the elements of $\mathcal{B}$.

Problem 7.9. *Given a finite family $\mathcal{X}$ of graphs, determine the growth rates of the maximum q-th weak coloring number and q-centered chromatic number of $\mathcal{X}$-minor-free graphs.*

One of the most tantalizing special case of this problem is $\mathcal{X} = \{K_5, K_{3,3}\}$, that is to determine the growth rates of weak coloring numbers and centered chromatic numbers of planar graphs. A first motivation for this particular case is the central role of planar graphs in graph theory. Moreover, we solved Problem 7.9 in Chapter 6 for every family $\mathcal{X}$ containing a planar graph. Therefore, the smallest minor-closed class of graphs with unknown weak coloring numbers and centered chromatic numbers growth rates is the class of planar graphs. For weak coloring numbers, the current best bounds are $\Omega(q^2 \log q)$ [JM22] and $\mathcal{O}(q^3)$ [vdHOQ+17]. For centered chromatic numbers, the gap is slightly bigger with $\Omega(q^2 \log q)$ and $\mathcal{O}(q^3 \log q)$ [DMSF21]. It was conjectured by Joret and Micek in [JM22] for weak coloring numbers, and by Dębski, Micek, Schröder, and Felsner in [DMSF21] for centered chromatic numbers, that the aforementioned lower bounds are optimal.

Conjecture 7.10 ([JM22, DMSF21])**.** *The maximum q-th weak coloring number and q-centered chromatic number of planar graphs are in $\Theta(q^2 \log q)$.*

Note that these lower bounds in $\Omega(q^2 \log q)$ are reached for the same construction which consists of planar graphs with treewidth at most 3 (namely graphs in $\mathcal{S}_4$, see Appendix A). Moreover, our results imply the upper bound $\mathcal{O}(q^2 \log q)$ for planar graphs of bounded treewidth. Therefore, any improvement in the lower bounds will require graphs of unbounded treewidth.

Another challenging line of research is to consider graphs that exclude a fixed topological minor. We say that a graph H is a *topological minor* of a graph G if there exists an injective function $\varphi\colon V(H) \to V(G)$ and a family $(P_{xy} \mid xy \in E(H))$ of pairwise internally disjoint paths in G such that the endpoints of P_{xy} are $\varphi(x)$ and $\varphi(y)$, for every $xy \in E(H)$. Note that every topological minor of a graph G is a minor of G. If a graph X is not a topological minor of a graph G, then we say that G is *X-topological-minor-free*.

Dębski, Micek, Schröder, and Felsner proved in [DMSF21] that for every graph X, there exists $f(X)$ such that graphs that do not have X as a topological minor have q-centered chromatic numbers in $\mathcal{O}(q^{f(X)})$. However, no explicit bound on $f(X)$ is known, and finding good estimations of $f(X)$, even when $X = K_t$, is a challenging open problem. I tend to believe that the upper bound $\mathcal{O}(q^{t-1})$ given by Theorem 1.32 for K_t-minor-free graphs also holds for K_t-topological-minor-free graphs.

Conjecture 7.11. *For every positive integer t, for every positive integer q, the maximum q-centered chromatic number of a K_t-topological-minor-free graph is in $\mathcal{O}(q^{t-1})$.*

Note that weak coloring numbers have a very different behavior in this setting. Indeed, graphs with maximum degree at most 3 have exponential weak coloring numbers [GKR+18], while they exclude $K_{1,4}$ as a topological minor. As a consequence, the similarity highlighted in Chapter 6 between weak coloring numbers and centered chromatic numbers does not survive in the topological minor setting.

The technique used in [DMSF21] to prove that graphs of fixed maximum degree Δ have q-centered chromatic number at most polynomial in q (and even linear in q) is entropy compression, and so is essentially probabilistic. This seems very far from the tools we developed in Chapter 6 for K_t-minor-free graphs. Since Conjecture 7.11 covers both graphs of bounded maximum degree and K_t-minor-free graphs, a positive answer would probably require to combine these apparently very different approaches. This problem was circumvent by Dębski, Micek, Schröder, and Felsner in [DMSF21] by using a decomposition theorem of Grohe and Marx [GM15], which asserts that, for every fixed positive integer t, there exists $f(t)$ such that K_t-topological-minor-free graphs admits tree decompositions in which the torso of every bags is either $K_{f(t)}$-minor-free, or is obtained from a graph of maximum degree at most $f(t)$ by adding at most $f(t)$ vertices. Combined with a lemma by Pilipczuk and Siebertz [PS21], this gives a polynomial bound for the q-centered chromatic numbers of K_t-topological-minor-free graphs, knowing such a polynomial bound for both $K_{f(t)}$-minor-free graphs and graphs of bounded maximum degree. However, this approach seems to fail to obtain more accurate bounds.

We finish this conclusion with a remark concerning the distinction between excluding one or several minors. The bounds exposed in Chapter 6 are essentially the same when excluding one or several minors. We conjecture that this is a general fact.

Conjecture 7.12. *Let* $\mathrm{par} \in \{\mathrm{cen}, \mathrm{wcol}\}$*, and let* $\mathcal{X}$ *be a nonempty finite family of graphs. There exists* $X \in \mathcal{X}$ *such that, for every positive integer q,*

$$\max_{G\ \mathcal{X}\text{-minor-free}} \mathrm{par}_q(G) = \Theta\left(\max_{G\ X\text{-minor-free}} \mathrm{par}_q(G)\right).$$

Again, one of the most intriguing open case is $\mathcal{X} = \{K_5, K_{3,3}\}$, which corresponds to planar graphs.

Bibliography

[BD02] Patrick Bellenbaum and Reinhard Diestel. Two short proofs concerning tree-decompositions. *Combinatorics, Probability and Computing*, 11(6):541–547, 2002. 40, 54

[BDJM20] Prosenjit Bose, Vida Dujmović, Mehrnoosh Javarsineh, and Pat Morin. Asymptotically optimal vertex ranking of planar graphs. *arXiv preprint*, 2020. arXiv:2007.06455. 24

[BRST91] Dan Bienstock, Neil Robertson, Paul Seymour, and Robin Thomas. Quickly excluding a forest. *Journal of Combinatorial Theory, Series B*, 52(2):274–283, 1991. 11, 99

[CC16] Chandra Chekuri and Julia Chuzhoy. Polynomial bounds for the Grid-Minor Theorem. *Journal of the ACM*, 63(5):1–65, 2016. arXiv:1305.6577. 220

[CKL+20] Marek Cygan, Paweł Komosa, Daniel Lokshtanov, Marcin Pilipczuk, Michał Pilipczuk, Saket Saurabh, and Magnus Wahlström. Randomized contractions meet lean decompositions. *ACM Transactions on Algorithms*, 17(1):1–30, 2020. arXiv:1810.06864. 56

[CRST06] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. The strong perfect graph theorem. *Annals of Mathematics*, 164(1):51–229, 2006. 3

[CT21] Julia Chuzhoy and Zihan Tan. Towards tight(er) bounds for the excluded grid theorem. *Journal of Combinatorial Theory, Series B*, 146:219–265, 2021. arXiv:1901.07944. 220

[DEG+21] Vida Dujmović, Louis Esperet, Cyril Gavoille, Gwenaël Joret, Piotr Micek, and Pat Morin. Adjacency labelling for planar graphs (and beyond). *Journal of the ACM*, 68(6):1–33, 2021. arXiv:2003.04280. 24

[DEJ+20a] Vida Dujmović, David Eppstein, Gwenaël Joret, Pat Morin, and David R. Wood. Minor-closed graph classes with bounded layered pathwidth. *SIAM Journal on Discrete Mathematics*, 34(3):1693–1709, 2020. arXiv:1810.08314. 22, 95

[DEJ+20b] Vida Dujmović, Louis Esperet, Gwenaël Joret, Bartosz Walczak, and David R. Wood. Planar graphs have bounded nonrepetitive chromatic number. *Advances in Combinatorics*, 2020. arXiv:1904.05269. 24

[DEMW23] Vida Dujmović, Louis Esperet, Pat Morin, and David R. Wood. Proof of the clustered hadwiger conjecture. In *2023 IEEE 64th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1921–1930, 2023. arXiv:2306.06224. 24

[DHH+24] Vida Dujmović, Robert Hickingbotham, Jędrzej Hodor, Gwenaël Joret, Hoang La, Piotr Micek, Pat Morin, Clément Rambaud, and David R. Wood. The Grid-Minor Theorem revisited. In *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1241–1245, 2024. arXiv:2307.02816. 24, 27, 111, 112, 165, 168, 221, 251

[DHJ+23] Vida Dujmović, Robert Hickingbotham, Gwenaël Joret, Piotr Micek, Pat Morin, and David R. Wood. The excluded tree minor theorem revisited. *Combinatorics, Probability and Computing*, 33(1):85–90, 2023. arXiv:2303.14970. 86, 88, 157, 197

[Die95] Reinhard Diestel. Graph Minors I: A short proof of the path-width theorem. *Combinatorics, Probability and Computing*, 4(1):27–30, 1995. 11, 99, 100

[Die17] Reinhard Diestel. *Graph Theory*. Springer Berlin, 2017. 3, 7, 205

[DJ24] David Dekker and Bart Jansen. Kernelization for feedback vertex set via elimination distance to a forest. *Discrete Applied Mathematics*, 346:192–214, 2024. arXiv:2206.04387. 219

[DJM+20] Vida Dujmović, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R. Wood. Planar graphs have bounded queue-number. *Journal of the ACM*, 67(4), 2020. arXiv:1904.04791. 23, 24

[DJMM24] Vida Dujmović, Gwenaël Joret, Piotr Micek, and Pat Morin. Tight bound for the Erdős-Pósa property of tree minors. *Combinatorics, Probability and Computing*, 34(2):321–325, 2024. arXiv:2403.06370. 109, 221

[DMSF21] Michał Dębski, Piotr Micek, Felix Schröder, and Stefan Felsner. Improved bounds for centered colorings. *Advances in Combinatorics*, 2021. arXiv:1907.04586v3. 24, 26, 27, 31, 33, 197, 198, 207, 222, 223, 233

[DMW17] Vida Dujmović, Pat Morin, and David R. Wood. Layered separators in minor-closed graph classes with applications. *Journal of Combinatorial Theory, Series B*, 127:111–147, 2017. arXiv:1306.1595. 21, 22, 24, 95, 111, 112, 129

[DS20] Zdeněk Dvořák and Jean-Sébastien Sereni. On fractional fragility rates of graph classes. *The Electronic Journal of Combinatorics*, 27(4), 2020. arXiv:1907.12634. 26, 27, 128, 159, 238

[Dvo13] Zdeněk Dvořák. Constant-factor approximation of the domination number in sparse graphs. *European Journal of Combinatorics*, 34(5):833–840, 2013. arXiv:1110.5190. 27

[Dvo16] Zdeněk Dvořák. Sublinear separators, fragility and subexponential expansion. *European Journal of Combinatorics*, 52:103–119, 2016. arXiv:1404.7219. 26

[EGK+17] Kord Eickmeyer, Archontia C. Giannopoulou, Stephan Kreutzer, O-joung Kwon, Michal Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. Neighborhood complexity and kernelization for nowhere dense classes of graphs. In *ICALP 2017*, volume 80 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 63:1–63:14, 2017. arXiv:1612.08197. 27

[EJM23] Louis Esperet, Gwenaël Joret, and Pat Morin. Sparse universal graphs for planarity. *Journal of the London Mathematical Society*, 108(4):1333–1357, 2023. arXiv:2010.05779. 24

[EP65] Paul Erdős and Lajos Pósa. On independent circuits contained in a graph. *Canadian Journal of Mathematics*, 17:347–352, 1965. 9

[Epp00] David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3):275–291, 2000. 20, 21, 22

[Erd18] Joshua Erde. A unified treatment of linked and lean tree-decompositions. *Journal of Combinatorial Theory, Series B*, 130:114–143, 2018. arXiv:1703.03756. 39

[ES35] Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935. 50

[FMW13] Ruy Fabila-Monroy and David R. Wood. Rooted K_4-minors. *The Electronic Journal of Combinatorics*, 20(2), 2013. arXiv:1102.3760. 20

[FN06] Pierre Fraigniaud and Nicolas Nisse. Connected treewidth and connected graph searching. In *Proceedings of the 7th Latin American Symposium on Theoretical Informatics, LATIN 2006*, volume 3887 of *Lecture Notes in Computer Science*, pages 479–490, 2006. 113

[Gal64] Tibor Gallai. Maximum-minimum sätze und verallgemeinerte faktoren von graphen. *Acta Mathematica Academiae Scientiarum Hungaricae*, 12(1–2):131–173, 1964. 221

[GJ16] Jim Geelen and Benson Joeris. A generalization of the Grid Theorem. *arXiv preprint*, 2016. arXiv:1609.09098. 219

[GJNW23] Carla Groenland, Gwenaël Joret, Wojciech Nadara, and Bartosz Walczak. Approximating pathwidth for graphs of small treewidth. *ACM Transactions on Algorithms*, 19(2), 2023. arXiv:2008.00779. 113

[GKR+18] Martin Grohe, Stephan Kreutzer, Roman Rabinovich, Sebastian Siebertz, and Konstantinos Stavropoulos. Coloring and covering nowhere dense graphs. *SIAM Journal on Discrete Mathematics*, 32(4):2467–2481, 2018. arXiv:1602.05926. 27, 28, 31, 34, 222, 231

[GKS17] Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *Journal of the ACM*, 64(3), 2017. arXiv:1311.3899. 27

[GM15] Martin Grohe and Dániel Marx. Structure theorem and isomorphism test for graphs with excluded topological subgraphs. *SIAM Journal on Computing*, 44(1):114–159, 2015. arXiv:1111.1109. 223

[Hal67] Rudolf Halin. Zur klassifikation der endlichen graphen nach H. Hadwiger und K. Wagner. *Mathematische Annalen*, 172(1):46–78, 1967. 17

[Hal76] Rudolf Halin. S-functions for graphs. *Journal of Geometry*, 8(1–2):171–186, 1976. 17

[vdHOQ+17] Jan van den Heuvel, Patrice Ossona de Mendez, Daniel Quiroz, Roman Rabinovich, and Sebastian Siebertz. On the generalised colouring numbers of graphs that exclude a fixed minor. *European Journal of Combinatorics*, 66:129–144, 2017. arXiv:1602.09052. 26, 27, 34, 222

[vdHW18] Jan van den Heuvel and David R. Wood. Improper colourings inspired by Hadwiger's conjecture. *Journal of the London Mathematical Society*, 98(1):129–148, 2018. arXiv:1704.06536. 27

[Hig52] Graham Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, s3-2(1):326–336, 1952. 219

[HJM+21] Tony Huynh, Gwenaël Joret, Piotr Micek, Michał T. Seweryn, and Paul Wollan. Excluding a ladder. *Combinatorica*, 42(3):405–432, 2021. arXiv:2002.00496. 12, 13, 19, 40, 61, 245

[HLMR24a] Jędrzej Hodor, Hoang La, Piotr Micek, and Clément Rambaud. Centered colorings in minor-closed graph classes. *arXiv preprint*, 2024. arXiv:2411.02122. 77, 125

[HLMR24b] Jędrzej Hodor, Hoang La, Piotr Micek, and Clément Rambaud. Quickly excluding an apex-forest. *arXiv preprint*, 2024. arXiv:2404.17306. 41, 77, 95, 105, 106

[HLMR25] Jędrzej Hodor, Hoang La, Piotr Micek, and Clément Rambaud. Weak coloring numbers of minor-closed graph classes. In *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, page 3325–3334, January 2025. arXiv:2407.04588. 77, 125

[ISW24] Freddie Illingworth, Alex Scott, and David R. Wood. Product structure of graphs with an excluded minor. *Transactions of the American Mathematical Society, Series B*, 11(35):1233–1248, 2024. arXiv:2104.06627. 111, 160, 182, 184

[JM22] Gwenaël Joret and Piotr Micek. Improved bounds for weak coloring numbers. *The Electronic Journal of Combinatorics*, 29(1), 2022. arXiv:2102.10061. 27, 34, 128, 165, 171, 222, 231, 232

[KPS24] Tuukka Korhonen, Michał Pilipczuk, and Giannos Stamoulis. Minor containment and disjoint paths in almost-linear time. In *2024 IEEE 65th Annual Symposium on Foundations of Computer Science (FOCS)*, page 53–61, 2024. arXiv:2404.03958. 7

[MSW17] Daniel Marx, Paul Seymour, and Paul Wollan. Rooted grid minors. *Journal of Combinatorial Theory, Series B*, 122:428–437, 2017. arXiv:1307.8138. 20, 84, 105, 106

[NOdM12] Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity*, volume 28 of *Algorithms and Combinatorics*. Springer Berlin, 2012. 25

[NOdMW12] Jaroslav Nešetřil, Patrice Ossona de Mendez, and David R. Wood. Characterisations and examples of graph classes with bounded expansion. *European Journal of Combinatorics*, 33(3):350–373, 2012. arXiv:0902.3265. 25

[OdMOW19] Patrice Ossona de Mendez, Sang-il Oum, and David R. Wood. Defective colouring of graphs excluding a subgraph or minor. *Combinatorica*, 39(2):377–410, 2019. arXiv:1611.09060. 24

[PPS20] Marcin Pilipczuk, Michał Pilipczuk, and Sebastian Siebertz. Lecture notes for the course “Sparsity” given at Faculty of Mathematics, Informatics, and Mechanics of the University of Warsaw, Winter semesters 2017/18 and 2019/20. https://www.mimuw.edu.pl/~mp248287/sparsity2. 25

[PPT23a] Christophe Paul, Evangelos Protopapas, and Dimitrios M. Thilikos. Graph parameters, universal obstructions, and WQO. *arXiv preprint*, 2023. arXiv:2304.03688. 218

[PPT23b] Christophe Paul, Evangelos Protopapas, and Dimitrios M. Thilikos. Universal obstructions of graph parameters. *arXiv preprint*, 2023. arXiv:2304.14121. 153

[PS21] Michał Pilipczuk and Sebastian Siebertz. Polynomial bounds for centered colorings on proper minor-closed graph classes. *Journal of Combinatorial Theory, Series B*, 151:111–147, 2021. arXiv:1807.03683. 26, 27, 33, 181, 223

[Ram97] Siddharthan Ramachandramurthi. The structure and number of obstructions to treewidth. *SIAM Journal on Discrete Mathematics*, 10(1):146–157, 1997. 219

[Ram24] Clément Rambaud. Excluding a rectangular grid. *arXiv preprint*, 2024. arXiv:2501.11617. 37

[Ree92] Bruce Reed. Finding approximate separators and computing tree width quickly. In *Proceedings of the twenty-fourth annual ACM symposium on Theory of computing*, STOC ’92, pages 221–228, 1992. 15

[RS] Neil Robertson and Paul Seymour. Graph width and well-quasi-ordering. Unpublished. 39

[RS83] Neil Robertson and Paul Seymour. Graph minors. I. Excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983. 11, 28, 99

[RS86a] Neil Robertson and Paul Seymour. Graph minors. II. Algorithmic aspects of tree-width. *Journal of Algorithms*, 7(3):309–322, 1986. 17

[RS86b] Neil Robertson and Paul Seymour. Graph minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114, 1986. 10, 15

[RS95] Neil Robertson and Paul Seymour. Graph minors. XIII. The disjoint paths problem. *Journal of Combinatorial Theory, Series B*, 63(1):65–110, 1995. 7, 20

[RS03] Neil Robertson and Paul Seymour. Graph ninors. XVI. Excluding a non-planar graph. *Journal of Combinatorial Theory, Series B*, 89(1):43–76, 2003. 26

[RS04] Neil Robertson and Paul Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, 2004. 7

[RS23] Felix Reidl and Blair D. Sullivan. A color-avoiding approach to subgraph counting in bounded expansion classes. *Algorithmica*, 85(8):2318–2347, 2023. arXiv:2001.05236. 27

[RST96] Neil Robertson, Paul Seymour, and Robin Thomas. Excluding infinite clique minors. *Memoirs of the American Mathematical Society*, 118(566):0–0, 1996. 217

[Sey23] Paul Seymour. A shorter proof of the path-width theorem. *arXiv preprint*, 2023. arXiv:2309.05100. 100

[Tho90] Robin Thomas. A Menger-like property of tree-width: The finite case. *Journal of Combinatorial Theory, Series B*, 48(1):67–76, 1990. 40, 54

[UWY22] Torsten Ueckerdt, David Wood, and Wendy Yi. An improved Planar Graph Product Structure Theorem. *The Electronic Journal of Combinatorics*, 29(2), 2022. arXiv:2108.00198. 24

[Wol08] Paul Wollan. Extremal functions for rooted minors. *Journal of Graph Theory*, 58(2):159–178, 2008. 20

APPENDIX A

Appendix to Chapter 6

In this appendix, we provide proofs for the already known lower bounds used in Chapter 6, as well as some properties of rooted 2-treedepth, and in particular its link with 2-treedepth.

A.1 Lower bounds

A.1.1 Weak coloring numbers

In this section, we present a construction by Grohe, Kreutzer, and Siebertz [GKR+18], which was initially presented as a lower bound for weak coloring numbers of graph of treewidth t. This was later adapted to graph of simple treewidth at most t by Joret and Micek [JM22]. To show the link between these constructions and the families $(\mathcal{R}_t \mid t \in \mathbb{N}_{>0})$, $(\mathcal{S}_t \mid t \in \mathbb{N}_{>0})$, we elected to provide them in the current appendix. The core of the argument is the following lemma.

Lemma A.1. *Let $\mathcal{G}$ be a nonempty class of graphs, and let $g(q) = \max_{G\in\mathcal{G}} \mathrm{wcol}_q(G)$ for every nonnegative integer q. For every nonnegative integer q, there exists $G \in \mathbf{T}(\mathcal{G})$ such that*

$$\mathrm{wcol}_q(G) \geqslant \sum_{i=0}^{q} g(i).$$

Proof. We proceed by induction on q. First suppose that $q = 0$. Then consider $G \in \mathcal{G}$ such that $\mathrm{wcol}_q(G) = g(0)$. Since $\mathcal{G} \subseteq \mathbf{T}(\mathcal{G})$, G is a as desired.

Now suppose $q \geqslant 1$. By the inductive hypothesis there exists $G_0 \in \mathbf{T}(\mathcal{G})$ with

$$\mathrm{wcol}_{q-1}(G_0) \geqslant \sum_{i=0}^{q-1} g(i).$$

Let $H \in \mathcal{G}$ be such that $\mathrm{wcol}_q(H) = g(q)$. Let $N = \sum_{i=0}^{q} g(i)$. Let G be a graph obtained from G_0 by adding for each $u \in V(G_0)$, N disjoint copies $H_{u,1}, \dots, H_{u,N}$ of H, and all possible edges between u and $V(H_{u,i})$, for every $i \in [N]$.

We claim that $G \in \mathbf{T}(\mathcal{G})$. Let $(T_0, (W_{0,a} \mid a \in V(T_0)))$ be a rooted forest decomposition of G_0 witnessing the fact that $G_0 \in \mathbf{T}(\mathcal{G})$. Then, for every $u \in V(G_0)$, choose $a_u \in V(T_0)$ such that $u \in W_{0,a_u}$. Let T be the rooted forest obtained from T_0 by adding, for every $u \in V(G_0)$ and for every $i \in [N]$, a leaf $a_{u,i}$ with parent a vertex a_u. Then, for every $a \in V(T)$, let

$$W_a = \begin{cases} W_{0,a} & \text{if } a \in V(T_0) \\ \{u\} \cup V(H_{u,i}) & \text{if } a = a_{u,i} \text{ for } u \in V(G_0), i \in [N]. \end{cases}$$

It follows that $(T, (W_a \mid a \in V(T)))$ is a rooted forest decomposition of G witnessing the fact that $G \in \mathbf{T}(\mathcal{G})$.

It remains to show that $\operatorname{wcol}_q(G) \geqslant N$. Let σ be an ordering of $V(G)$. Since $\operatorname{wcol}_{q-1}(G_0) \geqslant \sum_{i=0}^{q-1} g(i)$, there exists $v \in V(G_0)$ such that

$$|\operatorname{WReach}_{q-1}[G_0, \sigma|_{V(G_0)}, v]| \geqslant \sum_{i=0}^{q-1} g(i).$$

If for every $i \in [N]$, there is a vertex w in $V(H_{v,i})$ such that $w <_\sigma v$, then $|\operatorname{WReach}_q[G, \sigma, v]| \geqslant N$. Otherwise, there exists $i \in [N]$ such that every vertex w in $H_{v,i}$ is such that $w >_\sigma v$. Since $\operatorname{wcol}_q(H_{v,i}) = \operatorname{wcol}_q(H) = g(q)$, there exists $u \in V(H_{v,i})$ such that $|\operatorname{WReach}_q[H_{v,i}, \sigma|_{V(H_{v,i})}, u]| \geqslant g(q)$. Then,

$$\operatorname{WReach}_{q-1}[G_0, \sigma|_{V(G_0)}, v] \cup \operatorname{WReach}_q[H_{v,i}, \sigma|_{V(H_{v,i})}, u] \subseteq \operatorname{WReach}_q[G, \sigma|_{V(G)}, u],$$

and we conclude that $|\operatorname{WReach}_r[G, \sigma, u]| \geqslant \left(\sum_{i=0}^{q-1} g(i)\right) + g(q) = N$. □

Corollary A.2. *For every integer t with $t \geqslant 1$,*

$$\max_{G \in \mathcal{R}_t} \operatorname{wcol}_q(G) = \Omega(q^{t-1}).$$

Proof. We show by induction on t that for every nonnegative integer q,

$$\max_{G \in \mathcal{R}_t} \operatorname{wcol}_q(G) \geqslant \binom{q+t-1}{t-1}.$$

For $t = 1$, recall that $K_1 \in \mathcal{R}_1$. Therefore, for all nonnegative integers q we have $\operatorname{wcol}_q(K_1) = 1 \geqslant \binom{q}{0}$, as desired.

For $t \geqslant 2$, by the induction hypothesis, for every nonnegative integer q,

$$\max_{G \in \mathcal{R}_{t-1}} \operatorname{wcol}_q(G) \geqslant \binom{q+t-2}{t-2}.$$

Therefore, by Lemma A.1,

$$\max_{G \in \mathcal{R}_t} \operatorname{wcol}_q(G) \geqslant \sum_{i=0}^{q} \binom{q+t-2}{t-2} = \binom{q+t-1}{t-1}.$$ □

Corollary A.3. *For every positive integer t with $t \geqslant 2$,*

$$\max_{G \in \mathcal{S}_t} \operatorname{wcol}_q(G) = \Omega(q^{t-2} \log q).$$

Proof. We proceed by induction on t. For $t = 2$, $\mathcal{S}_2$ contains every path, which have q-th weak coloring number in $\Omega(\log q)$, see [JM22].

For $t \geqslant 3$, by the induction hypothesis, there exists $c > 0$ such that for every integer q with $q \geqslant 2$,

$$\max_{G \in \mathcal{S}_{t-1}} \operatorname{wcol}_q(G) \geqslant c \cdot q^{t-3} \log q.$$

Therefore, by Lemma A.1, for every integer q with $q \geqslant 4$,

$$\begin{aligned}\max_{G \in \mathcal{S}_t} \operatorname{wcol}_q(G) &\geqslant c \cdot \sum_{i=1}^{q} i^{t-3} \log i \\ &\geqslant c \cdot \sum_{q/2 \leqslant i \leqslant q} i^{t-3} \log i \\ &\geqslant c \cdot \left\lfloor \frac{q-1}{2} \right\rfloor (q/2)^{t-3} \log(q/2) \\ &\geqslant c \cdot \frac{q}{4} \cdot \frac{1}{2^{t-3}} \cdot q^{t-3} \cdot \frac{\log q}{2} \\ &\geqslant \frac{c}{2^t} \cdot q^{t-2} \log q. \end{aligned}$$

□

A.1.2 Centered colorings

In this section, we prove lower bounds for centered chromatic numbers of graphs in $\mathcal{R}_t$ and $\mathcal{S}_t$. This is a rewriting with our notations of the proof published in [DMSF21].

Let q, t be positive integers, let G be a graph. A mapping $\Psi\colon V(G) \to 2^{2^C}$ is said to be *t-good* if for every $u \in V(G)$, $\emptyset \in \Psi(u)$, and for every $U \subseteq C$, if there exists $Y \in \Psi(u)$ such that $U \subseteq Y$, then there exists $Y' \in \Psi(u)$ such that

(i) $U \subseteq Y'$, and

(ii) $|Y' \setminus U| \leqslant t$.

A pair $\psi = (\psi_1, \Psi_2)$ where $\psi_1\colon V(G) \to C$ and $\Psi_2\colon V(G) \to 2^{2^C}$, is a *$q$-centered generalized coloring* if for every connected subgraph H of G, for every $\psi_2 \in \prod_{u \in V(H)} \Psi_2(u)$[1], either

(i) $|\psi_1(V(H)) \cup \bigcup \psi_2(V(H))| > q$, or

(ii) there is a ψ-center of $V(H)$, that is a vertex $u \in V(H)$ such that $\psi_1(u) \notin \psi_1(V(H) \setminus \{u\}) \cup \bigcup \Psi_2(V(H))$.

The number of colors used by ψ is the integer $|\psi_1(V(H)) \cup \bigcup_{u \in V(G)} \bigcup \Psi_2(u)|$.

Now, we define $\operatorname{cen}_{q,t}^{(N)}(G)$ to be the minimum integer k, such that there exists a generalized q-centered coloring $\psi = (\psi_1, \Psi_2)$ of G using at most N colors such that

(i) Ψ_2 is t-good, and

(ii) for every $u \in V(G)$, there exists $Y \in \Psi_2(u)$ such that $|\psi_1(V(G)) \setminus Y| \leqslant k$.

By convention, we set $\operatorname{cen}_{q,t}^{(N)}(G) = \infty$ if there is no such integer k.

Lemma A.4. *Let q, N be positive integers. For every positive integer n with $n \leqslant 2q$,*

$$\operatorname{cen}_{4q,q}^{(N)}(P_n) \geqslant \lfloor \log n \rfloor.$$

[1] Given a set X and a family $(Y_x \mid x \in X)$ of sets, we denote by $\prod_{x \in X} Y_x$ the set of all the functions $f\colon X \to \bigcup_{x \in X} Y_x$ such that $f(x) \in Y_x$ for every $x \in X$.

Proof. It is enough to consider the case $n = 2^k$ for some nonnegative integer k. Let (ψ_1, Ψ_2) be a generalized $4q$-centered coloring of P_n with Ψ_2 q-good, and let U be the set of the colors of all the ψ_1-centers of $V(P_n)$. We will prove by induction on k that there exists $u \in V(P_n)$ such that

$$|U \setminus Y| \geqslant k$$

for every $Y \in \Psi_2(u)$. This is clear when $k = 0$. Now suppose $k \geqslant 1$.

Let U be the set of the colors of all the ψ_1-centers of $V(P_n)$. Let Q_1, Q_2 be the subpaths of P_n induced by, respectively, the first $n/2 = 2^{k-1}$ vertices and last $n/2 = 2^{k-1}$ vertices of P_n. Let $U_1 = U \cap V(Q_1)$ and $U_2 = U \cap V(Q_2)$. Note that U_1 and U_2 are disjoint.

Suppose there exists $u_1 \in V(Q_1)$ and $Y_1 \in \Psi_2(u_1)$ such that $U_2 \subseteq Y_1$, and $u_1 \in V(Q_1)$ and $Y_1 \in \Psi_2(u_1)$ such that $U_1 \subseteq Y_2$. Then, since Ψ_2 is q-good, there exists $Y_1' \in \Psi_2(u_1)$ and $Y_2' \in \Psi_2(u_2)$ such that $U_2 \subseteq Y_1'$, $U_1 \subseteq Y_2'$, and $|Y_1' \setminus U_2|, |Y_2' \setminus U_1| \leqslant q$. Then, let $\psi_2 \in \prod_{u \in V(P_n)} \Psi_2(u)$ be defined by

$$\psi_2(u) = \begin{cases} Y_1' & \text{if } u = u_1, \\ Y_2' & \text{if } u = u_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since (ψ_1, Ψ_2) is $4q$-centered, either

(i) $|\psi_1(V(P_n)) \cup \bigcup_{u \in V(H)} \psi_2(u)| > 4q$, but then $|V(P_n)| + |Y_1' \setminus U| + |Y_2' \setminus U| > 4q$, which implies $n > 2q$, a contradiction; or

(ii) there is a (ψ_1, ψ_2)-center u of $V(H)$. In particular, $\psi_1(u) \in U$. Without loss of generality, suppose $\psi_1(u) \in U_1$. But then, $\psi_1(u) \in Y_2' = \psi_2(u_2)$, contradicting the fact that u is (ψ_1, ψ_2)-center of $V(H)$.

This proves that there exists $a \in \{1, 2\}$ such that for every $u \in V(Q_a)$, for every $Y \in \Psi_2(u)$, $U_{3-a} \setminus Y \neq \emptyset$. We now call the induction hypothesis on Q_a, $\psi_1|_{V(Q_a)}$, $\Psi_2|_{V(Q_a)}$. Therefore, there exists $u \in V(Q_a)$ such that for every $Y \in \Psi_2(u)$,

$$|U_a \setminus Y| \geqslant k - 1.$$

Since U_a and U_{3-a} are disjoint, we deduce that for every $Y \in \Psi_2(u)$,

$$|U \setminus Y| \geqslant |U_a \setminus Y| + |U_{3-a} \setminus Y| \geqslant k.$$

This concludes the proof of the induction, and implies the lemma. □

Lemma A.5. *Let q, n be positive integers, let $\mathcal{G}$ be a nonempty class of graphs, and let $g(t) = \max_{G \in \mathcal{G}} \mathrm{cen}^{(N)}_{q,t}(G)$ for every nonnegative integer t. For every integer t with $0 \leqslant t \leqslant q$, there exists $G \in \mathbf{T}(\mathcal{G})$ such that*

$$\mathrm{cen}^{(N)}_{q,t}(G) \geqslant \sum_{i=t}^{q} g(i).$$

Proof. We proceed by induction on $q - t$. When $q = t$, consider $G \in \mathcal{G}$ such that $\mathrm{cen}^{(N)}_{q,q}(G) = g(q)$. Since $\mathcal{G} \subseteq \mathbf{T}(\mathcal{G})$, G is a as wanted. Now suppose $t \leqslant q - 1$.

By the induction hypothesis, there exists $G_0 \in \mathbf{T}(\mathcal{G})$ such that

$$\mathrm{cen}_{q,t+1}^{(N)}(G_0) \geqslant \sum_{i=t+1}^{q} g(i).$$

Moreover, by the definition of g, there exists $G' \in \mathcal{G}$ such that

$$\mathrm{cen}_{q,t}^{(N)}(G') = g(t).$$

Let G be the graph obtained from G_0 as follows. For every $u \in V(G_0)$, consider disjoint copies $G'_{u,i}$ of G' for $i \in [2^N+1]$, and add every possible edge between u and $\bigcup_{i=1}^{2^N+1} V(G'_{u,i})$.

First we show that $G \in \mathbf{T}(\mathcal{G})$. Let $(T_0, (W_{0,a} \mid a \in V(T_0)))$ be a rooted forest decomposition of G_0 witnessing the fact that $G_0 \in \mathbf{T}(\mathcal{G})$. Then, for every $u \in V(G_0)$, choose $a_u \in V(T_0)$ such that $u \in W_{0,a_u}$. Let T be the rooted forest obtained from T_0 by adding, for every $u \in V(G_0)$ and for every $i \in [2^N+1]$, a leaf $a_{u,i}$ with parent a vertex a_u. Then, for every $a \in V(T)$, let

$$W_a = \begin{cases} W_{0,a} & \text{if } a \in V(T_0) \\ \{u\} \cup V(G'_{u,i}) & \text{if } a = a_{u,i} \text{ for } u \in V(G_0),\ i \in [2^N+1]. \end{cases}$$

It follows that $(T, (W_a \mid a \in V(T)))$ is a rooted forest decomposition of G witnessing the fact that $G \in \mathbf{T}(\mathcal{G})$.

We now show that $\mathrm{cen}_{q,t}^{(N)}(G) \geqslant \sum_{i=t}^{q} g(i)$. Let (ψ_1, Ψ_2) be a generalized q-centered coloring of G using at most N colors. Let $u \in V(G)$. By the pigeonhole principle, there exists $i, j \in [2^N+1]$ such that $\psi_1(V(G'_{u,i})) = \psi_1(V(G'_{u,j}))$. Without loss of generality, assume $i = 1$ and $j = 2$. Since $\mathrm{cen}_{q,t}^{(N)}(G'_{u,1}) = g(t)$, and because $(\psi_1|_{V(G'_{u,1})}, \Psi_2|_{V(G'_{u,1})})$ is a generalized q-centered coloring of $G'_{u,1}$, there exists $v(u) \in V(G'_{u,1})$ such that

$$|\psi_1(V(G'_{u,1})) \setminus Y| \geqslant g(t)$$

for every $Y \in \Psi_2(v(u))$. Then let

$$\Psi_2^0(u) = \{\emptyset\} \cup \{X \cup Y \mid \psi_1(v(u)) \in X \subseteq \psi_1(V(G'_{u,1})), Y \in \Psi_2(v(u))\}.$$

Then, for every set of colors U, if $U \subseteq X \cup Y$ for some $X \subseteq \psi_1(G'_{u,1})$ with $\psi_1(V(U)) \in X$ and $Y \in \Psi_2(v(u))$, then, because Ψ_2 is t-good, there exists $Y' \in \Psi_2(v(u))$ such that $U \cap Y \subseteq Y'$ and $|Y' \setminus (U \cap Y)| \leqslant t$, and it follows that $Y^0 = (X \cap U) \cup \{\psi_1(v(u))\} \cup Y'$ is a member of $\Psi_2^0(v(u))$ such that $U \subseteq Y^0$ and $|Y^0 \setminus U| \leqslant t+1$. This proves that Ψ_2^0 is $(t+1)$-good.

Let $\psi_1^0 = \psi_1|_{V(G_0)}$. We claim that (ψ_1^0, Ψ_2^0) is a generalized q-centered coloring of G_0. Let H^0 be a connected subgraph of G_0 and let $\psi_2^0 \in \prod_{u \in V(H^0)} \Psi_2^0(u)$. Let H be a subgraph of G and $\psi_2 \in \prod_{u \in V(H)} \Psi_2(u)$ be obtained from H^0 as follows. Let $u \in V(H)$. Let $\psi_2(u) = \emptyset \in \Psi_2(u)$. If $\psi_2^0(u) \neq \emptyset$, then there exists $X \subseteq \psi_1(V(G'_{u,1}))$ with $\psi_1(v(u)) \in X$ and $Y \in \Psi_2(v(u))$ such that $\psi_2^0(u) = X \cup Y$. Since $\psi_1(V(G'_{u,1})) = \psi_1(V(G'_{u,2}))$, there exists $V_1 \subseteq V(G'_{u,1})$ and $V_2 \subseteq V(G'_{u,2})$ such that $\psi_1(V_1) = \psi_1(V_2) = X$, and $v(u) \in V_1$. Then, add the vertices in $V_1 \cup V_2$ to H. Moreover, for every $w \in V_1 \cup V_2$, add the edge uw and set

$$\psi_2(w) = \begin{cases} Y & \text{if } w = v(u), \\ \emptyset & \text{otherwise.} \end{cases}$$

This completes the description of H and ψ_2. Since (ψ_1, Ψ_2) is a generalized q-centered coloring of G, and because H is connected, either

(i) $|\psi_1(V(H)) \cup \bigcup \psi_2(V(H))| = |\psi_1^0(V(H^0)) \cup \bigcup \psi_2^0(V(H^0))| > q$, or

(ii) there is a (ψ_1, ψ_2)-center u of $V(H)$. We claim that u is a (ψ_1^0, ψ_2^0)-center of $V(H^0)$. Indeed, since every the color of every vertex in $V(H) \setminus V(H^0)$ is repeated at least twice, we have $u \in V(H^0)$. Moreover, $\bigcup \psi_2^0(V(H)) = \psi_1(V(H) \setminus V(H^0)) \cup \bigcup \psi_2(V(H))$. Therefore, $\psi_1^0(u) \notin \bigcup \psi_2^0(V(H))$. Since we also have $\psi_1^0(u) \notin \psi_1(V(H) \setminus \{u\})$, and because $\psi_1^0(V(H^0) \setminus \{u\}) \subseteq \psi_1(V(H) \setminus \{u\})$, we conclude that u is a (ψ_1^0, ψ_2^0)-center of $V(H^0)$.

This proves that (ψ_1^0, Ψ_2^0) is a generalized q-centered coloring of G_0.

Therefore, since $\mathrm{cen}_{q,t+1}^{(N)}(G_0) \geqslant \sum_{i=t+1}^{q} g(i)$, there exists $u \in V(G_0)$ such that, for every $Y^0 \in \Psi_2^0(u)$,

$$|\psi_1(V(G_0)) \setminus Y^0| \geqslant \sum_{i=t+1}^{q} g(i).$$

Therefore, for every $Y \in \Psi_2(v(u))$, apply this inequality for $Y^0 = \psi_1(V(G'_{u,1})) \cup Y$, we deduce that

$$|\psi_1(V(G_0)) \setminus (\psi_1(V(G'_{u,1})) \cup Y)| \geqslant \sum_{i=t+1}^{q} g(i).$$

Since we also have $|\psi_1(V(G'_{u,1})) \setminus Y| \geqslant g(t)$, we deduce

$$|\psi_1(V(G)) \setminus Y| \geqslant g(t) + \sum_{i=t+1}^{q} g(i) = \sum_{i=t}^{q} g(i).$$

This proves that $\mathrm{cen}_{q,t}^{(N)}(G) \geqslant \sum_{i=t}^{q} g(i)$ and concludes the proof of the lemma. □

Corollary A.6. *For every integer k with $k \geqslant 1$,*

$$\max_{G \in \mathcal{R}_k} \mathrm{cen}_q(G) = \Omega(q^{k-1}).$$

Proof. Let q, k be positive integers, and let $N = \binom{q+k-1}{k-1} - 1$. We show by induction on k that for every nonnegative integer t,

$$\max_{G \in \mathcal{R}_k} \mathrm{cen}_{q,t}^{(N)}(G) \geqslant \binom{q+k-1-t}{k-1}.$$

If $k = 1$, then either $t \geqslant 1$ and the result is clear, or $t = 0$ and $\binom{q+k-1-t}{k-1} = 1$. Assume we are in this later case. Consider $K_1 \in \mathcal{R}_1$, and let (ψ_1, Ψ_2) be a q-centered coloring of K_1 with Ψ_2 0-good. We denote by u the unique vertex of K_1. Then, consider $Y \in \Psi_2(u)$. If $\psi_1(u) \in Y$, then, because Ψ_2 is 0-good, we have $\{\psi_1(u)\} \in \Psi_2(u)$. But then, the subgraph K_1 of K_1 equipped with $\psi_2 \colon u \mapsto \{\psi_1(u)\}$ has no (ψ_1, ψ_2)-center but uses only $1 \leqslant q$ colors. This contradicts the fact that (ψ_1, Ψ_2) is a generalized q-centered coloring of K_1. Therefore, $\psi_1(u) \notin Y$, and so $|\psi_1(V(K_1)) \setminus Y| \geqslant 1$. This proves that $\mathrm{cen}_{q,0}^{(N)}(K_1) \geqslant 1$ and concludes the case $k = 1$. Now suppose $k > 1$.

By Lemma A.5, because $\mathcal{R}_k = \mathbf{T}(\mathcal{R}_{k-1})$, and applying the induction hypothesis, we deduce

$$\max_{G \in \mathcal{R}_k} \operatorname{cen}^{(N)}_{q,t}(G) \geqslant \sum_{i=t}^{q} \max_{G \in \mathcal{R}_{k-1}} \operatorname{cen}^{(N)}_{q,t}(G)$$
$$\geqslant \sum_{i=t}^{q} \binom{q+k-2-i}{k-2} = \binom{q+k-1-t}{k-1}.$$

This proves the induction.

Consider now a graph $G \in \mathcal{R}_k$ such that $\operatorname{cen}^{(N)}_{q,0}(G) \geqslant \binom{q+k-1}{k-1}$. Suppose for contradiction that G admits a q-centered coloring ψ using at most N colors. Then $(\psi, u \mapsto \emptyset)$ is a generalized q-centered coloring of G and $u \mapsto \emptyset$ is 0-good. Therefore, there exists $u \in V(G)$, and $Y \in \{\emptyset\}$ such that $|\psi(V(G)) \setminus Y| = |\psi(V(G))| \geqslant \binom{q+k-1}{k-1}$, contradicting the fact that ψ uses at most N colors. Therefore, every q-centered coloring of G uses at least $\binom{q+k-1}{k-1}$ colors. Since $\binom{q+k-1}{k-1} = \Omega(q^{k-1})$, this proves the corollary. □

Corollary A.7. *For every integer k with $k \geqslant 2$,*

$$\max_{G \in \mathcal{S}_k} \operatorname{cen}_q(G) = \Omega(q^{k-2} \log q).$$

Proof. Let q, k be positive integers, and let $N = \binom{q+k-2}{k-2}(\log(2q)-1)-1$. We show by induction on k that for every nonnegative integer t,

$$\max_{G \in \mathcal{S}_k} \operatorname{cen}^{(N)}_{4q,t}(G) \geqslant \binom{q+k-2-t}{k-2}(\log(2q)-1).$$

If $k = 2$, then $\binom{q+k-2-t}{k-2} \leqslant 1$, and this inequality follows from Lemma A.4. Now suppose $k > 2$.

By Lemma A.5, because $\mathcal{S}_k = \mathbf{T}(\mathcal{S}_{k-1})$, and applying the induction hypothesis, we deduce

$$\max_{G \in \mathcal{R}_k} \operatorname{cen}^{(N)}_{4q,t}(G) \geqslant \sum_{i=t}^{4q} \max_{G \in \mathcal{S}_{k-1}} \operatorname{cen}^{(N)}_{4q,t}(G)$$
$$\geqslant \sum_{i=t}^{q} \max_{G \in \mathcal{S}_{k-1}} \operatorname{cen}^{(N)}_{4q,t}(G)$$
$$\geqslant \sum_{i=t}^{q} \binom{q+k-3-i}{k-3}(\log(2q)-1)$$
$$= \binom{q+k-2-t}{k-2}(\log(2q)-1).$$

This concludes the induction.

Consider now a graph $G \in \mathcal{S}_k$ such that $\operatorname{cen}^{(N)}_{4q,0}(G) \geqslant \binom{q+k-2}{k-2}(\log(2q)-1)$. Suppose for contradiction that G admits a $4q$-centered coloring ψ using at most N colors. Then $(\psi, u \mapsto \emptyset)$ is a generalized $4q$-centered coloring of G and $u \mapsto \emptyset$ is 0-good. Therefore, there exists $u \in V(G)$, and $Y \in \{\emptyset\}$ such that $|\psi(V(G)) \setminus Y| = |\psi(V(G))| \geqslant \binom{q+k-2}{k-2}(\log(2q)-1) = N+1$, contradicting the fact that ψ uses at most N colors. Therefore, every $4q$-centered coloring of G uses at least $\binom{q+k-1}{k-1}(\log(2q)-1)$ colors. Since $\binom{\lfloor q/4 \rfloor+k-2}{k-2}(\log(\lfloor q/2 \rfloor)-1) = \Omega(q^{k-2}\log q)$, this proves the corollary. □

A.1.3 Fractional treedepth-fragility rates

In this section, we prove that for every integer t with $t \geqslant 2$, $\max_{G\in\mathcal{R}_t} \operatorname{ftdfr}_q(G) = \Omega(q^{t-1})$, and $\max_{G\in\mathcal{S}_t} \operatorname{ftdfr}_q(G) = \Omega(q^{t-2}\log q)$. The argument was sketched in [DS20], but we elected to provide a complete proof.

Let $\operatorname{td}(\cdot,\cdot)$ be defined as follows: for every graph G, if $\mathcal{C}$ denotes the family of all the connected components of G, then for every $t\colon V(G)\to\mathbb{N}$,

(i) $\operatorname{td}(G,t) = 0$ if $V(G)=\emptyset$,

(ii) $\operatorname{td}(G,t) = \max_{C\in\mathcal{C}} \operatorname{td}(C,t|_{V(C)})$, if $V(G)\neq\emptyset$ and G not connected, and

(iii) $\operatorname{td}(G,t) = 1+\max_{u\in V(G)}\max\{t(u),\operatorname{td}(G-u,t|_{V(G-u)})\}$ if G connected.

Lemma A.8. *Let G be a graph, let $t\colon V(G)\to\mathbb{N}$, let $U\subseteq V(G)$ with $G[U]$ connected, and let $v\in U$. If G' denotes the graph obtained from G by identifying the vertices in U into a single vertex w, and if $t'\colon V(G')\to\mathbb{N}$ is defined by $t'(w)=t(v)$ and $t'(x)=t(x)$ for every $x\in V(G)\setminus U$, then*

$$\operatorname{td}(G,t)\geqslant\operatorname{td}(G',t').$$

Proof. We proceed by induction on $|V(G)|$. The result is clear if $V(G)=U$. Now suppose $V(G)\setminus U\neq\emptyset$. If G is not connected, then consider the family $\mathcal{C}$ (resp. $\mathcal{C}'$) of all the connected components of G (resp. G'). Then $\operatorname{td}(G,t)=\max_{C\in\mathcal{C}}\operatorname{td}(C,t|_{V(C)})$. Let $C_0\in\mathcal{C}$ be the connected component of G containing U. This connected component corresponds to a connected component $C_0'\in\mathcal{C}'$ of G'. By the induction hypothesis, $\operatorname{td}(C_0,t|_{V(C_0)})\geqslant\operatorname{td}(C_0',t'|_{V(C_0')})$. Moreover, $\mathcal{C}\setminus\{C_0\}=\mathcal{C}'\setminus\{C_0'\}$. Therefore,

$$\operatorname{td}(G,t)=\max_{C\in\mathcal{C}}\operatorname{td}(C,t|_{V(C)})\geqslant\max_{C'\in\mathcal{C}'}\operatorname{td}(C',t'|_{V(C')})=\operatorname{td}(G',t').$$

Now suppose that G is connected. Therefore, there exists $z\in V(G)$ such that $\operatorname{td}(G,t)=1+\max\{t(z),\operatorname{td}(G-z,t|_{V(G-z)})\}$. First suppose $z\notin U$. Then, by the induction hypothesis, $\operatorname{td}(G-z,t|_{V(G-z)})\geqslant\operatorname{td}(G'-z,t'|_{V(G'-z)})$, and since $\operatorname{td}(G',t')\leqslant 1+\max\{t'(z),\operatorname{td}(G'-z,t'|_{V(G'-z)})\}$ and $t'(z)=t(z)$, we deduce that $\operatorname{td}(G,t)\geqslant\operatorname{td}(G',t')$. Now suppose $z\in U$. Then $G'-w\subseteq G-z$ and $t|_{V(G-u-v)}=t'|_{V(G'-w)}$, therefore $\operatorname{td}(G'-w,t'|_{V(G'-w)})\leqslant\operatorname{td}(G-z,t|_{V(G-z)})$. Moreover, if $z=v$, then $t(z)=t'(w)$, and if $z\neq v$, then $\operatorname{td}(G-z,t|_{G-z})\geqslant 1+t'(w)$. In both cases, $1+t'(w)\leqslant\max\{t(v),\operatorname{td}(G-z,t|_{G-z})\}$, and it follows that $\operatorname{td}(G,t)=1+\max\{t(z),\operatorname{td}(G-z,t|_{V(G-z)})\}\geqslant 1+\max\{t'(w),\operatorname{td}(G'-w,t|_{V(G'-w)})\}\geqslant\operatorname{td}(G',t')$. □

Lemma A.9. *Let G_0 be a graph, let $t_0\colon V(G_0)\to\mathbb{N}$, let $G_{u,a}$ for $u\in V(G_0)$ and $a\in\{1,2\}$ be a family of pairwise disjoint graphs, all of them disjoint from G_0. For all $u\in V(G_0)$ and $a\in\{1,2\}$, let $t_{u,a}\colon V(G)\to\mathbb{N}$. Let G be a graph with vertex set $V(G_0)\cup\bigcup_{u\in V(G_0),a\in\{1,2\}}V(G_{u,a})$ and such that $E(G_0)\subseteq E(G)$, $E(G_{u,a})\subseteq E(G)$, and $G[\{u\}\cup V(G_{u,a})]$ connected, for all $u\in V(G_0)$ and $a\in\{1,2\}$. For every $u\in V(G_0)$, let $t(u)=0$, and let $t(v)=t_{u,a}(v)$ for all $a\in\{1,2\}$ and $v\in V(G_{u,a})$. If $\operatorname{td}(G_{u,a},t_{u,a})\geqslant t_0(u)$ for all $u\in V(G_0)$ and $a\in\{1,2\}$, then*

$$\operatorname{td}(G,t)\geqslant\operatorname{td}(G_0,t_0).$$

Proof. For all $u\in V(G)$ and $a\in\{1,2\}$, for every $v\in V(G_{u,a})$, by replacing $t_{u,a}(v)$ by $\min\{t_{u,a}(v),t_0(u)\}$, we suppose $t(v)=t_{u,a}(v)\leqslant t_0(u)$.

We now proceed by induction on $|V(G)|$. If G is the null graph, then G_0 is null as well, and so $\mathrm{td}(G,t) = 0 = \mathrm{td}(G_0,t_0)$. Now suppose that G is nonnull.

Suppose now that G is not connected. Then G_0 is not connected. Let $\mathcal{C}_0$ be the family of all the connected components of G_0, and let $C_0 \in \mathcal{C}_0$. Let $G' = G[\bigcup_{u\in V(C_0)}(\{u\}\cup V(G_{u,1})\cup V(G_{u,2}))]$. Since G' is a subgraph of G, $\mathrm{td}(G,t) \geqslant \mathrm{td}(G', t|_{V(G')})$. Hence, by the induction hypothesis,

$$\mathrm{td}(G,t) \geqslant \mathrm{td}(G', t|_{V(G')}) \geqslant \mathrm{td}(C_0, t_0|_{V(C_0)}).$$

Therefore,

$$\mathrm{td}(G,t) \geqslant \max_{C_0\in\mathcal{C}_0} \mathrm{td}(C_0, t|_{V(C_0)}) = \mathrm{td}(G_0,t_0).$$

This concludes the case G not connected.

If G is connected, then there exists $v \in V(G)$ such that

$$\mathrm{td}(G,t) = 1 + \max\{t(v), \mathrm{td}(G-v, t|_{V(G-v)})\}.$$

Let $u \in V(G_0)$ and $a \in \{1,2\}$ such that $v \in V(G_{u,a}) \cup \{u\}$. Observe that $G_{u,3-a} \subseteq G - v$, and so $\mathrm{td}(G-v, t|_{V(G-v)}) \geqslant \mathrm{td}(G_{u,3-a}, t_{u,3-a}) \geqslant t_0(u) \geqslant t(v)$. In particular, $\mathrm{td}(G,t) = 1 + \mathrm{td}(G-v, t|_{V(G-v)})$. Moreover, by the induction hypothesis applied to $G' = G - (\{u\} \cup V(G_{u,1}) \cup V(G_{u,2}))$, we have $\mathrm{td}(G', t|_{G'}) \geqslant \mathrm{td}(G-u, t_0|_{G_0-u})$. Since $G-v$ contains both $G_{u,3-a}$ and G' as subgraphs, we conclude that

$$\begin{aligned}\mathrm{td}(G-v, t|_{V(G-v)}) &\geqslant \max\{\mathrm{td}(G_{u,3-a}, t_{u,3-a}), \mathrm{td}(G', t|_{V(G')})\}\\ &\geqslant \max\{t_0(u), \mathrm{td}(G_0-u, t|_{V(G_0-u)})\}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{td}(G,t) &= 1 + \mathrm{td}(G-v, t|_{V(G-v)})\\ &\geqslant 1 + \max\{t_0(u), \mathrm{td}(G_0-u, t|_{V(G_0-u)})\}\\ &\geqslant \mathrm{td}(G_0,t_0).\end{aligned}$$

This concludes the proof of the lemma. □

Let G be a graph, and let $w\colon V(G) \to \mathbb{Q}_{\geqslant 0}$. For every $U \subseteq V(G)$, we write $w(U) = \sum_{u\in U} w(u)$.

Let G be a graph and let q be a positive integer. We denote by $\mathrm{ftdfr}'_q(G)$ the maximum integer k such that there exists $w\colon V(G) \to \mathbb{Q}_{\geqslant 0}$ such that for every $\tau \in \mathbb{N}$, for every $t\colon V(G) \to \mathbb{N}$, for every $Y \subseteq V(G)$, if

(i) $w(Y) \leqslant \frac{1}{q} w(V(G))$, and

(ii) $w(\{u \in V(G) \mid t(u) < \tau\}) \leqslant \frac{1}{2} w(V(G))$,

then there exists $S_1, S_2 \subseteq V(G) \setminus Y$ disjoint such that

$$\mathrm{td}(G[S_a], t|_{S_a}) \geqslant \tau + k$$

for each $a \in \{1,2\}$.

First, we show that ftdfr'_q is a lower bound on ftdfr_q.

Lemma A.10. *For every graph G and for every positive integer q,*

$$\mathrm{ftdfr}_q(G) \geqslant \mathrm{ftdfr}'_q(G).$$

Proof. Let G be a graph, let q be a positive integer, and let Y be a random variable over subsets of $V(G)$ equipped with a q-thin distribution. Let $N = \mathrm{ftdfr}'_q(G)$ and let $w\colon V(G) \to \mathbb{Q}_{\geqslant 0}$ witnessing $\mathrm{ftdfr}'_q(G) \geqslant N$. Then, the expected value of $w(\mathsf{Y})$ is at most $\frac{w(V(G))}{q}$, and so there is a value $Y \subseteq V(G)$ of Y with $w(Y) \leqslant \frac{w(V(G))}{q}$. Then, considering $t\colon V(G) \to \mathbb{N}$ constant to 0, there exists $S_1, S_2 \subseteq V(G)$ disjoint with $\mathrm{td}(G[S_1], t|_{S_1}), \mathrm{td}(G[S_2], t|_{S_2}) \geqslant N$. In particular, $\mathrm{td}(G - Y) = \mathrm{td}(G - Y, t|_{V(G-Y)}) \geqslant N$. This proves that $\mathrm{ftdfr}_q(G) \geqslant N$. □

Now, we show that paths have ftdfr'_q at least logarithmic in q.

Lemma A.11. *For every positive integer q, there exists a positive integer ℓ such that*

$$\mathrm{ftdfr}'_q(P_\ell) \geqslant \log(q) - 13.$$

Proof. Let q be a positive integer. If $q \leqslant 2^{13}$, then the result is clear. Now suppose $q > 2^{13}$. Let $\ell' = \lfloor q/6 \rfloor$, let $\ell = 11\ell' + 1$, and let $N = \lfloor \log((\ell'/8) - 2) \rfloor + 1$. We label the vertices of P_ℓ by $0, \ldots, \ell - 1$, in this order. Let $w\colon V(P_\ell) \to \mathbb{Q}_{\geqslant 0}$ be constant to 1. Note that $w(V(P_\ell)) = \ell$. Let $Y \subseteq V(P_\ell)$ with $w(Y) \leqslant \frac{\ell}{q}$, let $t\colon V(P_\ell) \to \mathbb{N}$, and let τ be a nonnegative integer. Suppose that $w(\{u \in V(P_\ell) \mid t(u) \geqslant \tau\}) \geqslant \frac{\ell}{2}$.

Let i be a random variable over $\{-\ell' + 1, \ldots, \ell\}$ with uniform distribution, and let $\mathsf{I} = \{\mathsf{i}, \ldots, \mathsf{i} + \ell' - 1\} \cap \{0, \ldots, \ell - 1\}$. Then, the expected value of $|\{v \in \mathsf{I} \mid t(v) < \tau\}|$ is $\sum_{v \in V(P_\ell), t(u) < \tau} \Pr[u \in \mathsf{I}] \leqslant \frac{\ell}{2} \cdot \frac{\ell'}{\ell + \ell'} < \frac{\ell'}{2}$. In particular, by Markov's inequality,

$$\Pr\left[|\{v \in \mathsf{I} \mid t(v) < \tau\}| \geqslant \frac{3\ell'}{4}\right] \leqslant \frac{\ell'/2}{3\ell'/4} = \frac{2}{3}. \tag{A.1}$$

Moreover, $\Pr[\mathsf{I} \cap Y \neq \emptyset] \leqslant \sum_{u \in Y} \Pr[u \in \mathsf{I}] \leqslant \frac{\ell}{q} \cdot \frac{\ell'}{\ell} = \frac{\ell'}{q}$. Therefore,

$$\Pr[\mathsf{I} \cap Y \neq \emptyset] \leqslant \frac{1}{6}. \tag{A.2}$$

Additionally, we have

$$\Pr[|I| < \ell'] = \Pr[\mathsf{i} \in \{-\ell' + 1, \ldots, -1\} \cup \{\ell - (\ell' - 1), \ldots, \ell\}] = \frac{2\ell'}{\ell + \ell'} < \frac{1}{6} \tag{A.3}$$

Therefore, there exists a value I of I such that none of the events in (A.1), (A.2), and (A.3), are satisfied. In other words, I is such that $I \cap Y = \emptyset$, $|\{v \in I \mid t(v) < \tau\}| < \frac{3\ell'}{4}$, and $|I| = \ell'$. As a consequence, $Q = P_\ell[I]$ is a path on ℓ' vertices with at least $\frac{\ell'}{4}$ vertices v with $t(v) \geqslant \tau$. Let Q_1, Q_2 be two disjoint subpaths of Q, both having at least $\frac{\ell'}{8} - 1$ vertices v with $t(v) \geqslant \tau$. Let $a \in \{1, 2\}$. Let $Q' = P_{\lfloor \ell'/8 - 1 \rfloor}$, and let $t'\colon V(Q') \to \mathbb{N}$ constant to τ. By Lemma A.8, $\mathrm{td}(Q_a, t|_{V(Q)}) \geqslant \mathrm{td}(Q', t')$ for each $a \in \{1, 2\}$. Now, a simple induction shows that

$$\mathrm{td}(Q', t') \geqslant \min_{v \in V(Q)} t'(v) + \lfloor \log(|V(Q')| - 1) \rfloor + 1,$$

and so $\mathrm{td}(Q_a, t|_{V(Q)}) \geqslant \tau + \lfloor \log((\ell'/8) - 2) \rfloor + 1 = \tau + N$. Since Y, t, τ were arbitrary, this proves that

$$\mathrm{ftdfr}'_q(P_\ell) \geqslant N \geqslant \log((q - 102)/48) \geqslant \log(q/(48 \times 102)) \geqslant \log q - 13.$$ □

Next, we show that applying the operator $\mathbf{T}(\cdot)$ morally multiplies by $\Omega(q)$ the value of ftdfr'_q.

Lemma A.12. *Let $\mathcal{G}$ be a nonempty class of graphs, and let $g(q) = \max_{G \in \mathcal{G}} \mathrm{ftdfr}'_q(G)$ for every positive integer q. For every positive integer q, there exists $G \in \mathbf{T}(\mathcal{G})$ such that*

$$\mathrm{ftdfr}'_{4q}(G) \geqslant \sum_{i=1}^{q} g(2i).$$

Proof. Let q be a positive integer. For every $i \in [q]$, let G'_i be a graph in $\mathcal{G}$ with $\mathrm{ftdfr}'_q(G'_i) = g(2i)$, and let $w'_i \colon V(G'_i) \to \mathbb{Q}_{\geqslant 0}$ witnessing this fact. For every $i \in [q]$, by possibly removing from G'_i every vertex v with $w'_i(v) = 0$, we assume that w'_i takes only positive values. Let G' be the disjoint union of G'_i for $i \in [q]$. Let $w' \colon V(G') \to \mathbb{Q}_{\geqslant 0}$ be defined by $w'(u) = w_i(u)$ for every $u \in V(G'_i)$, for every $i \in [q]$. Note that $w'(v) > 0$ for every $v \in V(G')$, and $\mathrm{ftdfr}'_{2i}(G') \geqslant \mathrm{ftdfr}'_{2i}(G'_i) \geqslant g(2i)$ for every $i \in [q]$, as witnessed by w'. By possibly replacing w' by $\frac{1}{w'(V(G'))} \cdot w'$, we assume $w'(V(G')) = 1$.

Let h be a positive integer, and let G_h be the graph with vertex set

$$\bigcup_{i=0}^{h} V(G')^i$$

and edge set

$$\bigcup_{i=1}^{h} \Big(\Big\{ (u_1, \dots, u_{i-1})(u_1, \dots, u_i) \mid (u_1, \dots, u_i) \in V(G')^i \Big\} \\ \cup \Big\{ (u_1, \dots, u_{i-1}, v)(u_1, \dots, u_{i-1}, w) \mid (u_1, \dots, u_{i-1}) \in V(G')^{i-1}, vw \in E(G') \Big\} \Big).$$

See Figure A.1. For every $u = (u_1, \dots, u_i) \in V(G_h)$, we define the depth of u to be the integer i.

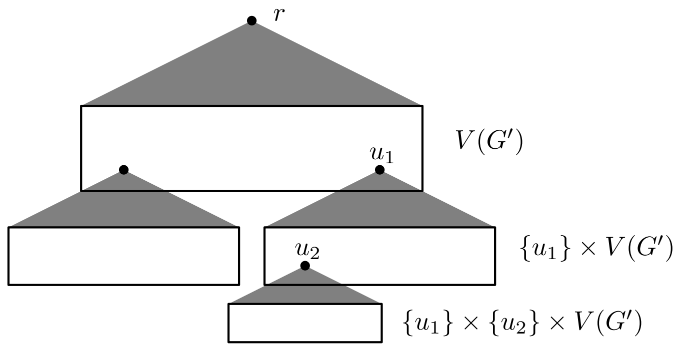


Figure A.1: Illustration for the definition of G_h.

We denote by r the unique vertex of G_h of depth 0. Then, for every $u = (u_1, \dots, u_i) \in V(G_h)$, let

$$w(u) = \prod_{j=1}^{i} w'(u_i).$$

In particular, $w(r) = 1$. Note that $w'(V(G_h)) = h + 1$. For every $u = (u_1, \dots, u_i) \in V(G_h)$, we denote by $G_{h,u}$ the subgraph of G_h induced by $\{(v_1, \dots, v_j) \mid i \leqslant j \leqslant h, (v_1, \dots, v_j) \in V(G), (v_1, \dots, v_i) = (u_1, \dots, u_i)\}$. Note that $G_{h,u}$ is isomorphic to G_{h-i}. The neighbors of u in $G_{h,u}$ are called the children of u in G_h.

Claim A.12.1. *For every positive integer h, G_h is in $\mathbf{T}(\mathcal{G})$.*

Proof of the claim. Let h be a positive integer. Let $\mathcal{G}'$ be the class of all the graphs G such that every connected component of G is in $\mathcal{G}$. Note that $\mathbf{T}(\mathcal{G}') = \mathbf{T}(\mathcal{G})$. Therefore, it is enough to show that $G_h \in \mathbf{T}(\mathcal{G}')$. Note that $G' \in \mathcal{G}'$.

Let T be the tree defined by

$$V(T) = \bigcup_{i=0}^{h-1} V(G')^i$$

and

$$E(T) = \bigcup_{i=1}^{h-1} \left(\left\{(u_1,\dots,u_{i-1})(u_1,\dots,u_i) \mid (u_1,\dots,u_i) \in V(G')^i\right\}\right).$$

Let $r \in V(T)$ corresponding to the empty tuple. We root T in r. For every $x = (x_1,\dots,x_i) \in V(T)$, let

$$W_x = \{x\} \cup \{(x_1,\dots,x_i,y) \mid y \in V(G')\}.$$

Then, the pair $(T,(W_x \mid x \in V(T)))$ is a tree decomposition of G witnessing the fact that $G_h \in \mathbf{T}(\mathcal{G}')$. ♢

Claim A.12.2. *Let q be a positive integer, let h be a positive integer, let $s \in V(G)$, let $S \subseteq V(G_{h,s})$ such that $s \in S$ and $G[S]$ connected, let $t\colon S \to \mathbb{N}$, and let $\tau \in \mathbb{N}$. If $w(\{u \in S \mid t(u) \geqslant \tau\}) > 2q \cdot w(s)$, then there exists $S_0, S_1, S_2 \subseteq S$ disjoint such that,*

(a) *$s \in S_0$;*

(b) *$G_h[S_a]$ is connected for every $a \in \{0,1,2\}$;*

(c) *there is an edge between S_0 and S_a in G_h, for every $a \in \{1,2\}$; and*

(d) *$\operatorname{td}(G_h[S_a], t|_{S_a}) \geqslant \tau + \sum_{i=1}^{q} g(2i)$ for every $a \in \{1,2\}$.*

Proof of the claim. Let d be the depth of s in G. We proceed by induction on $q + (h - d)$.

If $d = h$, then $S = \{s\}$, and so $w(\{u \in S \mid t(u) \geqslant \tau\}) \leqslant w(s)$. It follows that the claim is vacuously true. Now suppose $d < h$ and $w(\{u \in S \mid t(u) \geqslant \tau\}) > 2q \cdot w(s)$.

Let A be the set of all the children of s in G_h, and let $B = A \cap S$. For every $v \in B$, let $S_v = S \cap V(G_{h,v})$. Note that for every $v \in B$, S_v induces a connected subgraph of G_h. If there exists $v \in B$ such that $w(\{u \in S_v \mid t(u) \geqslant \tau\}) > 2q \cdot w(v)$, then, since the depth of v is larger than the depth of s, we can call the induction hypothesis on S_v. This gives three disjoint sets $S_0^0, S_1^0, S_2^0 \subseteq S_v$ satisfying (a)-(d). Then $(S_0, S_1, S_2) = (S_0 \cup \{s\}, S_1, S_2)$ are as wanted. Now assume

$$\forall v \in B, w(\{u \in S_v \mid t(u) \geqslant \tau\}) \leqslant 2q \cdot w(v). \tag{A.4}$$

We deduce from (A.4) that

$$\begin{aligned} 2q \cdot w(s) &< w(\{u \in S \mid t(u) \geqslant \tau\}) \\ &\leqslant w(s) + \sum_{v \in B} w(\{v \in S_v \mid t(u) \geqslant \tau\}) \\ &\leqslant w(s) + \sum_{v \in B} 2q \cdot w(v) \\ &\leqslant w(s) + 2q \cdot w(B). \end{aligned}$$

Therefore, since $w(s) = w(A)$, this implies

$$w(B) \geqslant \left(1 - \frac{1}{2q}\right) \cdot w(A). \tag{A.5}$$

Let $C = \{v \in B \mid w(\{u \in S_v \mid t(u) \geqslant \tau\}) \leqslant 2(q-1) \cdot w(v)\}$. Using (A.4), we deduce

$$\begin{aligned} 2q \cdot w(s) &< w(\{u \in S \mid t(u) \geqslant \tau\}) \\ &\leqslant w(s) + \sum_{v \in C} w(\{u \in S_v \mid t(u) \geqslant \tau\}) + \sum_{v \in B \setminus C} w(\{u \in S_v \mid t(u) \geqslant \tau\}) \\ &\leqslant w(s) + 2(q-1) \cdot w(C) + 2q \cdot w(B \setminus C) \\ &= w(s) + 2q \cdot w(B) - 2w(C) \\ &\leqslant (2q+1) \cdot w(s) - 2w(C). \end{aligned}$$

Therefore, since $w(s) = w(A)$,

$$w(C) \leqslant \frac{w(A)}{2}. \tag{A.6}$$

For every $v \in B \setminus C$, we have $w(\{u \in S_v \mid t(u) \geqslant \tau\}) > 2(q-1) \cdot w(v)$. If $q = 1$, then we deduce that, for every $v \in B \setminus C$, there exists $w \in S_v$ with $t(w) \geqslant \tau$. Since $G[A]$ is isomorphic to G' via the isomorphism $\varphi\colon A \to V(G')$ defined by $\varphi((s_1, \dots, s_d, w)) = w$ for every $w \in V(G')$, we can apply the definition of $\mathrm{ftdfr}'_2(G')$ with $Y' = \varphi(A \setminus B)$ and $t'\colon V(G') \to \mathbb{N}$ defined by $t'(w) = \tau$ for every $w \in \varphi(B \setminus C)$ and $t'(w) = 0$ for every $w \in \varphi(C)$. Then $w'(\{w \in V(G') \mid t'(w) \geqslant \tau\}) \geqslant \frac{w'(V(G'))}{2}$, and $w'(Y) \leqslant \frac{1}{2q} w'(V(G'))$ by (A.5). Therefore, there exists $S'_1, S'_2 \subseteq V(G' - Y)$ disjoint with $\mathrm{td}(G'[S'_a], t'|_{S'_a}) \geqslant \tau + g(2)$ for every $a \in \{1, 2\}$. Without loss of generality, $G'[S'_1]$ and $G'[S'_2]$ are connected. Then, let $(S_0, S_1, S_2) = (\{s\}, \bigcup_{w \in S'_1} S_{\varphi^{-1}(w)}, \bigcup_{w \in S'_2} S_{\varphi^{-1}(w)})$. Now applying Lemma A.9 to the graph obtained from $G_h[S]$ by contracting S_w into a single vertex for each $w \in B \setminus C$, and applying Lemma A.8, we deduce $\mathrm{td}(G_h[S_a], t|_{S_a}) \geqslant \tau + g(2)$ for every $a \in \{1, 2\}$. This concludes the case $q = 1$. Now suppose $q \geqslant 2$.

Let $v \in B \setminus C$. Since $w(\{u \in S_v \mid t(u) \geqslant \tau\}) > 2(q-1)w(v)$, by the induction hypothesis, there exists $S_{v,0}, S_{v,1}, S_{v,2} \subseteq S_v \setminus \{v\}$ disjoint such that

(a') $v \in S_{v,0}$;

(b') $G_h[S_{v,a}]$ connected for every $a \in \{0, 1, 2\}$;

(c') there is an edge between $S_{v,0}$ and $S_{v,a}$ in G_h, for every $a \in \{1, 2\}$; and

(d') $\mathrm{td}(G_h[S_{v,a}], t|_{S_{v,a}}) \geqslant \tau + \sum_{i=1}^{q-1} g(2i)$ for every $a \in \{1, 2\}$.

Moreover, $G[A]$ is isomorphic to G', and w' witnesses the fact that $\mathrm{ftdfr}'_{2q}(G[A]) \geqslant g(2q)$. By (A.5), the set $Y = A \setminus B$ satisfies $w(Y) \leqslant \frac{1}{2q} w(A)$, and so $w'(Y) \leqslant \frac{1}{2q} w'(A)$. Moreover, for $\tau' = \tau + \sum_{i=1}^{q-1} g(2i)$ and $t'\colon V(A) \to \mathbb{N}$ defined by

$$t'(v) = \begin{cases} \tau' & \text{if } v \in C, \\ 0 & \text{otherwise,} \end{cases}$$

we have $w(\{u \in B \mid t'(u) \geqslant \tau'\}) = w(C) \geqslant \frac{1}{2}w(A)$ by (A.6), and so $w'(\{u \in B \mid t'(u) \geqslant \tau'\}) \geqslant \frac{1}{2}w'(A)$. Since $G[A]$ is isomorphic to G' via the isomorphism $\varphi\colon A \to V(G')$ defined by $\varphi((s_1, \dots, s_d, w)) = w$ for every $w \in V(G')$, we can apply the definition of $\mathrm{ftdfr}'_{2q}(G')$ with $Y' = \varphi(A \setminus B)$ and $t'\colon V(G') \to \mathbb{N}$ defined by $t'(w) = \tau'$ for every $w \in \varphi(B \setminus C)$ and $t'(w) = 0$ for every $w \in \varphi(C)$. there exists $S'_1, S'_2 \subseteq A$ disjoint such that

$$\mathrm{td}(G_h[S'_a], t|_{S'_a}) \geqslant \tau' + g(2q) = \tau + \sum_{i=0}^{q} g(2i)$$

for each $a \in \{1, 2\}$.

Let $a \in \{1, 2\}$. Let S_a be the union of S'_a with $S_{v,1} \cup S_{v,2}$ for every $v \in S'_a \cap C$. Note that S_1, S_2 induces connected subgraphs of G_h. Then, let $S_0 = \{s\}$. Note that (a)-(c) hold by construction. It remains to prove that $\mathrm{td}(G[S_a], t|_{S_a}) \geqslant \tau + \sum_{i=0}^{q} g(2i)$ for every $a \in \{1, 2\}$. This follows from Lemma A.9 applied to the graph obtained from G by contracting $S_{v,0}$ for each $v \in C$, and by Lemma A.8. ♢

We now exhibit a graph in $\mathbf{T}(\mathcal{G})$ with $\mathrm{ftdfr}'_{4q}(G) \geqslant \sum_{i=1}^{q} g(2i)$. We take $G = G_h$ for $h = 4q$. Let $\tau \in \mathbb{N}$, let $t\colon V(G) \to \mathbb{N}$, and let $Y \subseteq V(G)$ such that

(i) $w(Y) \leqslant \frac{1}{2q}w(V(G))$, and

(ii) $w(\{u \in V(G) \mid t(u) < \tau\}) \leqslant \frac{1}{2}w(V(G))$.

For every $y \in Y \cup \{r\}$, let S_y be the vertex set of the connected component of y in $G_{h,y}[V(G_{h,y} - (Y \cup \{r\})) \cup \{y\}]$. Note that $\bigcup_{y \in Y \cup \{r\}} S_y = V(G)$, and so

$$w(V(G)) \leqslant \sum_{y \in Y \cup \{r\}} \frac{w(S_y)}{w(y)} \cdot w(y).$$

Therefore, there exists $y \in Y \cup \{r\}$ such that $\frac{w(S_y)}{w(y)} \geqslant \frac{w(V(G))}{w(Y \cup \{r\})}$, and so

$$w(S_y) \geqslant \frac{w(V(G))}{w(r) + w(Y)} \cdot w(y) \geqslant \frac{h+1}{1 + \frac{h+1}{4q}} \cdot w(y) > 2q \cdot w(y).$$

Therefore, by Claim A.12.2, there exist $S_1, S_2 \subseteq S_y \setminus \{y\}$ disjoint such that

$$\mathrm{td}(G[S_a], t|_{S_a}) \geqslant \tau + \sum_{i=1}^{q} g(2i)$$

for each $a \in \{1, 2\}$. Then, the sets S_1, S_2 are as wanted in the definition of $\mathrm{ftdfr}'(G)$. Since Y, t, τ were arbitrary, this shows that $\mathrm{ftdfr}'_{4q}(G) \geqslant \sum_{i=1}^{q} g(2i)$. □

We can now deduce the desired lower bounds on the growth rates of $(\mathrm{ftdfr}_q \mid q \in \mathbb{N}_{>0})$ in $\mathcal{R}_t$ and $\mathcal{S}_t$.

Corollary A.13. *For every integer k with $k \geqslant 1$,*

$$\max_{G \in \mathcal{R}_k} \mathrm{ftdfr}_q(G) = \Omega(q^{k-1}).$$

Proof. Let k be a positive integer. We show by induction on k that there exists $c > 0$ such that

$$\max_{G \in \mathcal{R}_k} \mathrm{ftdfr}'_q(G) \geqslant c \cdot q^{k-1}$$

for every positive integer q.

For $k = 1$, this is clear for $G = K_1$ and $c = 1$. Now suppose $k \geqslant 2$.

By the induction hypothesis, there exists $c' > 0$ such that $\max_{G \in \mathcal{R}_k} \mathrm{ftdfr}'_q(G) \geqslant c' \cdot q^{k-2}$ for every positive integer q. Let q be a positive integer. Without loss of generality, we suppose $q \geqslant 16$. Since $\mathcal{R}_k = \mathbf{T}(\mathcal{R}_{k-1})$, by Lemma A.12,

$$\begin{aligned}\max_{G \in \mathcal{R}_k} \mathrm{ftdfr}'_q(G) &\geqslant \sum_{i=1}^{\lfloor q/4 \rfloor} c' \cdot (2i)^{k-2} \\ &\geqslant c' \cdot \lfloor \lfloor q/4 \rfloor /2 \rfloor (2 \lfloor \lfloor q/4 \rfloor /2 \rfloor)^{k-2} \\ &\geqslant c' \cdot (q/16)(q/8)^{k-2} \\ &= \frac{c'}{2^{3k-2}} \cdot q^{k-1},\end{aligned}$$

which shows that $c = \frac{c'}{2^{3k-2}}$ is as wanted. By Lemma A.10, this proves the corollary. □

Corollary A.14. *For every integer k with $k \geqslant 2$,*

$$\max_{G \in \mathcal{S}_k} \mathrm{cen}_q(G) = \Omega(q^{k-2} \log q).$$

Proof. Let k be an integer with $k \geqslant 2$. We show by induction on k that there exists $c > 0$ such that

$$\max_{G \in \mathcal{S}_k} \mathrm{ftdfr}'_q(G) \geqslant c \cdot q^{k-2} \log q$$

for every integer q with $q \geqslant 2$. When $k = 2$, this is a consequence of Lemma A.11. Now suppose $k \geqslant 3$.

By the induction hypothesis, there exists $c' > 0$ such that $\max_{G \in \mathcal{R}_k} \mathrm{ftdfr}'_q(G) \geqslant c' \cdot q^{k-3} \log q$ for every integer q with $q \geqslant 2$. Let q be an integer with $q \geqslant 2$. Without loss of generality, we suppose $q \geqslant 16$. Since $\mathcal{S}_k = \mathbf{T}(\mathcal{S}_{k-1})$, by Lemma A.12,

$$\begin{aligned}\max_{G \in \mathcal{S}_k} \mathrm{ftdfr}'_q(G) &\geqslant \sum_{i=1}^{\lfloor q/4 \rfloor} c' \cdot (2i)^{k-3} \log(2i) \\ &\geqslant c' \cdot \lfloor \lfloor q/4 \rfloor /2 \rfloor (2 \lfloor \lfloor q/4 \rfloor /2 \rfloor)^{k-3} \log(2 \lfloor \lfloor q/4 \rfloor /2 \rfloor) \\ &\geqslant c' \cdot (q/16)(q/8)^{k-3} \log(q/4) \\ &\geqslant \frac{c'}{2^{3k-2}} \cdot q^{k-2} \log q,\end{aligned}$$

which shows that $c = \frac{c'}{2^{3k-2}}$ is as wanted. By Lemma A.10, this proves the corollary. □

A.2 2-treedepth and rooted 2-treedepth

In this section, we give a few facts about rooted 2-treedepth. In particular, we show that it is tied with the original notion of 2-treedepth, denoted by $\mathrm{td}_2(\cdot)$, as introduced by Huynh, Joret, Micek, Seweryn, and Wollan [HJM+21], which is defined by induction as follows. For every graph G, (t1) $\mathrm{td}_2(G) = 0$ if G is the null graph,

(t2) $\mathrm{td}_2(G) = \max_B \mathrm{td}_2(B)$ over all the blocks[2] B of G, if G has more than one block, and

(t3) $\mathrm{td}_2(G) = 1 + \min_{u \in V(G)} \mathrm{td}_2(G - u)$ if G has exactly one block.

We now give a similar definition of rooted 2-treedepth based on the following observations.

Observation A.15. *Let G be a graph. For every separation (A, B) of G of order at most one,*

$$\mathrm{rtd}_2(G) \leqslant \max\{\,\mathrm{rtd}_2(A), |V(A) \cap V(B)| + \mathrm{rtd}_2(B \setminus V(A))\}.$$

Proof. Consider a separation (A, B) of G of order at most one. Let $t = \max\{\,\mathrm{rtd}_2(A), \mathrm{rtd}_2\,(B \setminus V(A)) + |V(A) \cap V(B)|\}$. Suppose that G has at least one edge, and so $t \geqslant 2$. First consider the case (A, B) of order 0. By the definition of $\mathrm{rtd}_2(\cdot)$, $A, B \in \mathcal{R}_t$. Let $(F_1, (W_{1,x} \mid x \in V(F_1)))$, respectively $(F_2, (W_{2,x} \mid x \in V(F_2)))$, be a rooted forest decomposition of A (respectively B) witnessing the fact that $A \in \mathcal{R}_t = \mathbf{T}(\mathcal{R}_{t-1})$. Then $(F_1 \cup F_2, (W_x \mid x \in V(F_1 \cup F_2)))$, where $W_x = W_{i,x}$ for every $x \in V(F_i)$, for every $i \in \{1, 2\}$, is a rooted forest decomposition of G witnessing the fact that $G \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$. Now suppose that (A, B) has order one. Let the be a unique vertex in $V(A) \cap V(B)$. By the definition of $\mathrm{rtd}_2(\cdot)$, $A \in \mathcal{R}_t$ and $B \in \mathcal{R}_{t-1}$. Let $(F, (W_x \mid x \in V(F)))$ be a rooted forest decomposition of A witnessing the fact that $A \in \mathcal{R}_t = \mathbf{T}(\mathcal{R}_{t-1})$. There exists $y \in V(F)$ such that $u \in W_x$. Let F' be obtained from F by adding a fresh leaf z with parent y. Then, let

$$W'_x = \begin{cases} W_x & \text{if } x \in V(F), \\ V(B) & \text{if } x = z. \end{cases}$$

It follows that $(F', (W'_x \mid x \in V(F')))$ is a rooted forest decomposition witnessing the fact that $G \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$. This proves the observation. □

Observation A.16. *Let G be a graph with at least two vertices. There is a separation (A, B) of G of order at most one with $V(A) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$ such that*

$$\mathrm{rtd}_2(G) \geqslant \max\{\,\mathrm{rtd}_2(A), |V(A) \cap V(B)| + \mathrm{rtd}_2(B - V(A))\}.$$

Proof. Let t be a positive integer, and let $G \in \mathcal{R}_t$ with at least two vertices. We show that there exists a separation (A, B) of G of order at most one such that $A \in \mathcal{R}_t$ and $B \in \mathcal{R}_{t-|V(A) \cap V(B)|}$. Let $(T, (W_x \mid x \in V(T)))$ be a rooted forest decomposition of G witnessing the fact that $G \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$. Consider such a tree decomposition with $|V(T)|$ minimal. In particular, $W_x \neq \emptyset$ for every $x \in V(T)$ and $W_x \not\subseteq W_y$ for every $x, y \in V(T)$ adjacent.

If T is not a connected, then consider S a connected component of T. Let $A = G[\bigcup_{x \in V(S)} W_x]$ and $B = G - V(A)$. Then (A, B) is a separation of G of order 0, $(S, (W_x \mid x \in V(S)))$ witnesses the fact that $S \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$, and $(T - V(S), (W_x \mid x \in V(T - V(S))))$ witnesses the fact that $B \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$. Now suppose that T is connected.

If T has only one vertex, then $V(G)$ has at most one vertex, a contradiction. Hence T has at least two vertices. Therefore, T has a leaf y. Let $A = G[\bigcup_{x \in V(T) \setminus \{y\}} W_x]$ and $B = G[W_y]$. Since W_y is not included in $W_{\mathrm{p}(T,y)}$, $V(B) \setminus V(A) \neq \emptyset$, and since $W_{\mathrm{p}(T,y)} \neq \emptyset$, $V(A) \neq \emptyset$. Then $B - V(A) \in \mathcal{R}_{t-1}$ by the choice of $(T, (W_x \mid x \in V(T)))$. Moreover, $(T - y, (W_x \mid x \in V(T - y)))$ witnesses the fact that $A \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$. Since (A, B) has order one, this proves the observation. □

[2] A *block* of a graph G is a maximal subgraph of G which is either 2-connected, a single edge, or an isolated vertex.

Observations A.15 and A.16 yields the following explicit inductive definition of rooted 2-treedepth: for every graph G,

(r1) $\mathrm{rtd}_2(G) = 0$ if G is the null graph,

(r2) $\mathrm{rtd}_2(G) = 1$ if G is a one vertex graph, and otherwise

(r3) $\mathrm{rtd}_2(G)$ is the minimum of $\max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B \setminus V(A)) + |V(A) \cap V(B)|\}$ over all separations (A, B) of G of order at most one with $V(A) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$.

The following properties are direct consequences of this definition. For every graph G,

(r4) $\mathrm{rtd}_2(G) = \max_C \mathrm{rtd}_2(C)$ over all connected components C of G when G is not connected;

(r5) $\mathrm{rtd}_2(G) = \min_{v \in V(G)} \mathrm{rtd}_2(G - v) + 1$ when G consists of one block;

(r6) $\mathrm{rtd}_2(G) = \min_{(A,B)} \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B \setminus V(A)) + 1\}$ over all separations (A, B) of G of order one with $V(A) \cap V(B)$ consisting of a cut-vertex, when G is connected and consists of more than one block;

(r7) $\mathrm{rtd}_2(G) \leqslant 1 + \mathrm{rtd}_2(G - v)$ for every $v \in V(G)$;

Note that (r4) and (r7) imply by induction that $\mathrm{rtd}_2(G) \leqslant \mathrm{td}(G)$ for every graph G. Item (r3) in the definition can be in fact strengthened in the following way. For every graph G,

(r8) $\mathrm{rtd}_2(G)$ is the minimum of $\max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B \setminus V(A)) + |V(A) \cap V(B)|\}$ over all separations (A, B) of G of order at most one with $V(A) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$ such that B is a block.

Indeed, suppose that G is a graph and $\mathrm{rtd}_2(G) = \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B \setminus V(A)) + |V(A) \cap V(B)|\}$ where (A, B) is a separation of G of order at most one with $V(A) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$. Let (A', B') be a separation of G of order at most one with $V(A') \neq \emptyset$ and $V(B') \setminus V(A') \neq \emptyset$ such that B' is a block and $A \subseteq A'$ and $B' \subseteq B$. By definition $\mathrm{rtd}_2(G) \leqslant \max\{\mathrm{rtd}_2(A'), \mathrm{rtd}_2(B' \setminus V(A'))\}$. To prove the reverse inequality note that $\mathrm{rtd}_2(A') \leqslant \mathrm{rtd}_2(G)$ and $\mathrm{rtd}_2(B' - V(A')) \leqslant \mathrm{rtd}_2(B - V(A))$.

Lemma A.17. *For every graph G with at least one edge,*

$$\mathrm{td}_2(G) \leqslant \mathrm{rtd}_2(G) \leqslant 2\,\mathrm{td}_2(G) - 2.$$

Proof. First, we prove that $\mathrm{td}_2(G) \leqslant \mathrm{rtd}_2(G)$ for every graph G. We proceed by induction on $|V(G)|$. When G is a null graph, we have $\mathrm{td}_2(G) = \mathrm{rtd}_2(G) = 0$ and when G is a one-vertex graph, we have $\mathrm{td}_2(G) = \mathrm{rtd}_2(G) = 1$. Thus, we assume that $|V(G)| \geqslant 2$. If G consists of one block, then by (r5) and induction hypothesis,

$$\mathrm{td}_2(G) = \min_{v \in V(G)} \mathrm{td}_2(G - v) + 1 \leqslant \min_{v \in V(G)} \mathrm{rtd}_2(G - v) + 1 = \mathrm{rtd}_2(G).$$

If G consists of blocks $B_1, \ldots, B_k$ for $k > 1$, then by the induction hypothesis,

$$\mathrm{td}_2(G) = \max_{i \in [k]} \mathrm{td}_2(B_i) \leqslant \max_{i \in [k]} \mathrm{rtd}_2(B_i) \leqslant \mathrm{rtd}_2(G).$$

Now, we prove the other inequality for every graph G with at least one edge. We again proceed by induction on $|V(G)|$. If $\mathrm{td}_2(G) = 2$, then G is a forest with at least one edge, and so as mentioned earlier $\mathrm{rtd}_2(G) = \mathrm{td}_2(G) = 2$. Now assume that $\mathrm{td}_2(G) \geqslant 3$, and so in particular $|V(G)| \geqslant 3$, and that the result holds for smaller graphs. In particular, for every graph H with $|V(H)| < |V(G)|$, either H has no edge and so $\mathrm{rtd}_2(H) = \mathrm{td}_2(H) = 1$, or $\mathrm{rtd}_2(H) \leqslant 2\,\mathrm{td}_2(H)-2$. In both cases, $\mathrm{rtd}_2(H) \leqslant \max\{1, 2\,\mathrm{td}_2(H)-2\}$. By (r8), there is a separation (A, B) of G of order at most one such that $\mathrm{rtd}_2(G) = \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B - V(A)) + |V(A) \cap V(B)|\}$, $V(B) \setminus V(A) \neq \emptyset$, $V(A) \neq \emptyset$, and B is a block of G. If $|V(A) \cap V(B)| = 0$, then $B - V(A) = B$, and so

$$\begin{aligned}
\mathrm{rtd}_2(G) &= \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B)\} \\
&\leqslant \max\{\max\{1, 2\,\mathrm{td}_2(A) - 2\}, \max\{1, 2\,\mathrm{td}_2(B) - 2\}\} \\
&= \max\{1, 2\max\{\mathrm{td}_2(A), \mathrm{td}_2(B)\} - 2\} \\
&= 2\,\mathrm{td}_2(G) - 2.
\end{aligned}$$

Therefore, we assume that $|V(A) \cap V(B)| = 1$ and $V(A) \cap V(B) = \{u\}$. There exists $v \in V(B)$ such that $\mathrm{td}_2(B - v) = \mathrm{td}_2(B) - 1$. Then, by (r7),

$$\begin{aligned}
\mathrm{rtd}_2(B - u) &\leqslant \mathrm{rtd}_2(B - u - v) + 1 \leqslant \mathrm{rtd}_2(B - v) + 1 \\
&\leqslant \max\{1, 2\,\mathrm{td}_2(B - v) - 2\} + 1 \\
&= \max\{2, 2\,\mathrm{td}_2(B - v) - 1\} \\
&\leqslant \max\{2, 2\,\mathrm{td}_2(B) - 3\}.
\end{aligned}$$

Finally, since $\mathrm{td}_2(G) \geqslant 3$,

$$\begin{aligned}
\mathrm{rtd}_2(G) &= \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B \setminus u) + 1\} \\
&\leqslant \max\{\max\{1, 2\,\mathrm{td}_2(A) - 2\}, \max\{2, 2\,\mathrm{td}_2(B) - 3\} + 1\} \\
&= \max\{3, 2\,\mathrm{td}_2(A) - 2, 2\,\mathrm{td}_2(B) - 3 + 1\} \\
&\leqslant \max\{3, 2\,\mathrm{td}_2(G) - 2\} \\
&= 2\,\mathrm{td}_2(G) - 2.
\end{aligned}$$

□

The bounds in Lemma A.17 are tight. Indeed, for every positive integer n, we have $\mathrm{td}_2(K_n) = \mathrm{rtd}_2(K_n) = n$, which witnesses that the first inequality is tight. For the second one, see Lemma A.19, which we precede with a simple observation. Note that this observation is also true for td_2, namely, $\mathrm{td}_2(K_1 \oplus G) = 1 + \mathrm{td}_2(G)$ – again, the proof is very similar and we omit it.

Observation A.18. *For every graph G,*

$$\mathrm{rtd}_2(K_1 \oplus G) = 1 + \mathrm{rtd}_2(G).$$

Proof. Let G be a graph and let s the vertex of K_1 in $K_1 \oplus G$. By definition, $\mathrm{rtd}_2(K_1 \oplus G) \leqslant 1 + \mathrm{rtd}_2(G)$. For the other inequality, we proceed by induction on $\mathrm{rtd}_2(G)$. The assertion is clear when G is the null graph, thus, assume that G is not the null graph. If G is not connected, then $\mathrm{rtd}_2(G) = \mathrm{rtd}_2(C)$ for some connected component C of G, and since $\mathrm{rtd}_2(K_1 \oplus G) \geqslant \mathrm{rtd}_2(K_1 \oplus C)$, it suffices to show $\mathrm{rtd}_2(K_1 \oplus C) \geqslant 1 + \mathrm{rtd}_2(C)$. Therefore, we assume that G is connected. Since $K_1 \oplus G$ is also connected, there is a separation (A, B) of $K_1 \oplus G$ of order one such that $\mathrm{rtd}_2(G) = \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B \setminus V(A)) + 1\}$, $V(B) \setminus V(A) \neq \emptyset$, and $V(A) \neq \emptyset$. Since s is adjacent to all other vertices in $K_1 \oplus G$, the only possibility is that $V(A) = \{s\}$ and $V(B) = V(K_1 \oplus G)$. It follows that $B - V(A)$ contains a subgraph isomorphic to G, and thus, $\mathrm{rtd}_2(K_1 \oplus G) \geqslant 1 + \mathrm{rtd}_2(B \setminus V(A)) \geqslant 1 + \mathrm{rtd}_2(G)$. □

Lemma A.19. *For every integer k with $k \geqslant 2$, there is a graph G with $\mathrm{td}_2(G) \leqslant k$ and $\mathrm{rtd}_2(G) \geqslant 2k - 2$.*

Proof. We define inductively graphs $H_{k,\ell}$ with two distinguished vertices $u_{k,\ell}$ and $v_{k,\ell}$ for every integers k, ℓ with $k, \ell \geqslant 2$. For $k = 2$, $H_{k,\ell}$ is a path on ℓ vertices and $u_{k,\ell}, v_{k,\ell}$ are its endpoints. For $k \geqslant 3$, $H_{k,\ell}$ is obtained from two disjoint copies H_1, H_2 of $K_1 \oplus H_{k-1,\ell}$ by identifying the copy of $v_{k-1,\ell}$ in H_1 with the copy of $u_{k-1,\ell}$ in H_2. The vertices $u_{k,\ell}, v_{k,\ell}$ are then respectively the copy of $u_{k-1,\ell}$ in H_1 and the copy of $v_{k-1,\ell}$ in H_2. See Figure A.2.

By induction on k, we show that $\mathrm{td}_2(H_{k,\ell}) \leqslant k$ and $\mathrm{rtd}_2(H_{k,\ell}) \geqslant 2k - 2$ for all integers k, ℓ with $\ell \geqslant k \geqslant 2$. When $k = 2$, $H_{2,\ell}$ is a path on at least two vertices and so $\mathrm{td}_2(H_{2,\ell}) = \mathrm{rtd}_2(H_{2,\ell}) = 2$. Now suppose that $k \geqslant 3$. First, observe that $H_{k,\ell}$ has exactly two blocks H_1, H_2, both isomorphic to $K_1 \oplus H_{k-1,\ell}$. Hence, $\mathrm{td}_2(H_{k,\ell}) \leqslant \mathrm{td}_2(K_1 \oplus H_{k-1,\ell}) \leqslant 1 + \mathrm{td}_2(H_{k-1,\ell}) \leqslant k$ by the induction hypothesis. Let v be the unique cut-vertex of $H_{k,\ell}$. Since $H_{k,\ell}$ is connected and consists of more than one block, by (r6), there is a separation (A, B) of $H_{k,\ell}$ such that $V(A) \cap V(B) = \{v\}$ and $\mathrm{rtd}_2(H_{k,\ell}) = \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B - v) + 1\}$. It follows that the graph $B - v$ contains $K_1 \oplus H_{k-1,\ell-1}$ as a subgraph, and so, applying Observation A.18,

$$\mathrm{rtd}_2(H_{k,\ell-1}) \geqslant \mathrm{rtd}_2(B - v) + 1 \geqslant \mathrm{rtd}_2(K_1 \oplus H_{k-1,\ell-1}) + 1 \geqslant \mathrm{rtd}_2(H_{k-1,\ell-1}) + 2 \geqslant 2k - 2.$$

This concludes the proof of the lemma. □

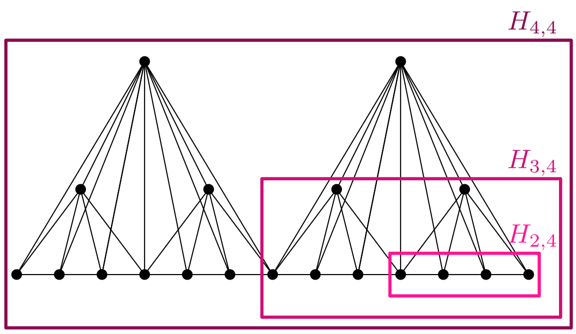


Figure A.2: The proof of Lemma A.19 implies that $\mathrm{td}_2(H_{4,4}) \leqslant 4$ and $\mathrm{rtd}_2(H_{4,4}) \geqslant 6$.

APPENDIX B

Excluding a rooted complete graph

In this chapter, we provide a proof of Theorem 7.7. This is a slight modification of Lemma 14 in [DHH+24].

Theorem 7.7 ([DHH+24, Lemma 14])**.** *Let t be a positive integer, let G be a graph, and let $S \subseteq V(G)$. If there is no S-rooted model of K_t in G, then there exists $S' \subseteq V(G)$ with $S \subseteq S'$ such that $\mathrm{torso}_G(S')$ is K_{2t-1}-minor-free.*

Lemma B.1. *Let t be a positive integer, let G be a graph, and let $S \subseteq V(G)$. Let $\mathcal{M} = (U_x \mid x \in V(K_{2t}))$ be a model of K_{2t} in $G \cup \binom{S}{2}$ such that $|\{x \in V(K_{2t}) \mid U_x \cap S \neq \emptyset\}| \leqslant t$. Then one of the following holds:*

(1) *there is an S-rooted model of K_t in G, or*

(2) *there is a separation (A, B) of G of order at most $t-1$ such that $S \subseteq V(A)$, and there exists $z \in V(K_{2t})$ such that $U_z \subseteq V(B) \setminus V(A)$.*

Proof. Suppose for contradiction that $G, S, \mathcal{M}$ is a counterexample with $|V(G)|$ minimum.

Claim B.1.1. *There is no separation (A, B) of G of order at most t such that $S \subseteq V(A)$, $V(A) \setminus S \neq \emptyset$, and $U_z \subseteq V(B) \setminus V(A)$ for some $z \in V(K_t)$.*

Proof of the claim. Suppose such a separation (A, B) exists, and let z be as in the statement. Since $G, S, \mathcal{M}$ is a counterexample of the lemma, there is no separation (A', B') of G of order at most $t-1$ with $S \subseteq V(A')$ and $V(B) \subseteq B'$. Therefore, by Menger's theorem, there is a family $(P_u \mid u \in V(A) \cap V(B))$ of pairwise disjoint $(S, V(A) \cap V(B))$-paths in G with $u \in V(P_u)$ for every $u \in V(A) \cap V(B)$. Note that, for all $u \in V(A) \cap V(B)$, P_u is a path in A. Let $G' = B$, let $S' = V(A) \cap V(B)$, and for every $x \in V(K_{2t})$, let $U'_x = U_x \cap V(B)$. Then, $\mathcal{M}' = (U'_x \mid x \in V(K_{2t}))$ is a model of K_{2t} in $G' \cup \binom{S'}{2}$. Since $|V(G')| < |V(G)|$, we can apply the minimality of $|V(G)|$.

In the first case, there is an S'-rooted model $(V_x \mid x \in V(K_t))$ of K_t in G'. But then, $(V_x \cup \bigcup_{u \in V_x \cap V(A) \cap V(B)} V(P_u) \mid x \in V(K_t))$ is an S-rooted model of K_t in G.

In the second case, there is a separation (A', B') of B with $S' \subseteq V(A)$ and $U'_{z'} \subseteq V(B') \setminus V(A')$ for some $z' \in V(K_t)$. But then we have $U'_{z'} = U_{z'}$, and $(A \cup A', B')$ is a separation of G of order at most $t-1$, with $S \subseteq V(A \cup A')$ and $U_{z'} \subseteq V(B') \setminus V(A \cup A')$.

In both cases, $G, S, \mathcal{M}$ is not a counterexample of the lemma. $\diamondsuit$

Claim B.1.2. *For every $x \in V(K_{2t})$, either $U_x \subseteq S$, or $|U_x| = 1$.*

Proof of the claim. Assume otherwise, and let y such that $U_y \not\subseteq S$ and $|U_y| > 1$. In particular, $G[U_y]$ has an edge uv with $\{u, v\} \not\subseteq S$. Let $G' = G/uv$, let w be the vertex in G' resulting from the identification of u and v, let

$$S' = \begin{cases} (S \setminus \{u, v\}) \cup \{w\} & \text{if } \{u, v\} \cap S \neq \emptyset \\ S & \text{otherwise,} \end{cases}$$

and for every $x \in V(K_{2t})$, let

$$U'_x = \begin{cases} (U_x \setminus \{u, v\}) \cup \{w\} & \text{if } \{u, v\} \cap U_x \neq \emptyset, \\ U_x & \text{otherwise.} \end{cases}$$

Then, since $|V(G')| < |V(G)|$, we can apply the minimality of $|V(G)|$.

In the first case, G' has an S'-rooted model of K_t, and we deduce that G has an S-rooted model of K_t.

In the second case, there is a separation (A', B') of G' of order at most $t-1$ with $U'_z \subseteq V(B') \setminus V(B')$ for some $z \in V(K_{2t})$. Let (A, B) be the separation of G obtained from (A', B') by uncontracting the edge uv. Note that (A, B) has order at most t. If (A, B) has order at most $t-1$, this contradicts the fact that $G, S, \mathcal{M}$ is not a counterexample of the lemma. Now suppose (A, B) has order exactly t, and so $u, v \in V(A)$. In particular, $V(A) \setminus S \neq \emptyset$. Since we also have $S \subseteq V(A)$ and $U_z \subseteq V(B) \setminus V(A)$, this contradicts Claim B.1.1. ♢

Let $T = \bigcup_{x \in V(K_{2t}), U_x \cap S = \emptyset} U_x$. By Claim B.1.2, T induces a complete graph in G, which has size at least t since $|\{x \in V(K_{2t}) \mid U_x \cap S \neq \emptyset\}| \leqslant t$. By Menger's theorem, either there is a separation (A, B) of G of order at most $t-1$ with $S \subseteq V(A)$ and $T \subseteq V(B)$, and so there exists $z \in T \setminus (V(A) \cap V(B))$, which implies $U_z \subseteq V(B) \setminus V(B)$; or there are t pairwise disjoint (S, T)-paths $Q_1, \ldots, Q_t$ in G, and then, $(V(Q_i) \mid i \in [t])$ is an S-rooted model of K_t in G. In both cases, we contradict the fact that $G, S, \mathcal{M}$ is a counterexample. □

Corollary B.2. *Let G be a connected graph, let t be a positive integer, and let S be a set of at least t vertices of G. If K_{2t} is a minor of $G \cup \binom{S}{2}$, then, for some $\ell \in [t]$, there is a separation (A, B) in G of order ℓ such that $S \subseteq V(A)$, and B contains a $(V(A) \cap V(B))$-rooted model of K_ℓ.*

Proof. We proceed by induction on t. For $t = 1$, one can take A to be a 1-vertex graph containing the vertex of S, and $B = G$. Note that $V(B) \setminus V(A)$ is non empty, and since G is connected, there is a vertex in $V(B) \setminus V(A)$ adjacent to the vertex in S. This vertex constitutes a $(V(A) \cap V(B))$-rooted model of K_1. Now, assume that $t \geqslant 2$ and that the result holds for all positive integers less than t. Let $\mathcal{M} = (U_x \mid x \in V(K_{2t}))$ be a model of K_{2t} in $G \cup \binom{S}{2}$. Apply Lemma B.1 to $G, S, \mathcal{M}$. If there is an S-rooted model of K_t in G, then take A to be the graph with vertex set S with no edges and B to be the whole graph G, and the lemma is satisfied with $\ell = t$. Otherwise, there exists a separation (C, D) in G of order at most $t-1$ and $z \in V(K_{2t})$ such that $S \subseteq V(C)$ and $U_z \subseteq V(D) \setminus V(C)$. Let E be the connected component of D containing U_z. Since z is adjacent to every other vertex in K_{2t}, U_x contains a vertex of E for every $x \in V(K_{2t})$. Let $\mathcal{M}_E$ be obtained from $\mathcal{M}$ by replacing each branch set in $\mathcal{M}$ by its intersection with $V(E)$. Let $S' = V(C) \cap V(E)$. Observe that $\mathcal{M}_E$ is a model of K_{2t} in $E \cup \binom{S'}{2}$. By the induction hypothesis applied to $E, S', \mathcal{M}_E$, there exists a separation (A', B') of order at most $t-1$ of E

such that $S' \subseteq V(A')$ and B' has a $(V(A') \cap V(B'))$-rooted model of $K_{|V(A') \cap V(B')|}$. Finally, put $A = C \cup A' \cup (D - V(E))$ and $B = B'$. It follows that the separation (A, B) has the desired properties. □

We can now deduce Theorem 7.7.

Proof of Theorem 7.7. Suppose that there is no S-rooted model of K_t in G. We proceed by induction on $|V(G)|$. If G is not connected, then we apply the induction hypothesis on each connected component and we are done. Now suppose G connected. If there is no model of K_{2t} in G, then we are done. Otherwise, let $\mathcal{M} = (U_x \mid x \in V(K_{2t}))$ be a model of K_{2t} in G. By Lemma B.1, there is a separation (A, B) of G with $S \subseteq V(A)$ and $U_z \subseteq V(B) \setminus V(A)$ for some $z \in V(K_{2t})$. Then, since z is adjacent to every other vertex in K_{2t}, the family $(U_x \cap V(B) \mid x \in V(K_{2t}))$ is a model of K_{2t} in $B \cup \binom{V(A) \cap V(B)}{2}$. By Corollary B.2, there is a separation (C, D) of B with $V(C) \cap V(D) \neq \emptyset$, $V(A) \cap V(B) \subseteq V(C)$, and such that there is a $(V(C) \cap V(D))$-rooted model of $K_{|V(C) \cap V(D)|}$ in D. In particular, this implies that $G' = \mathrm{torso}_G(V(C) \cup V(A))$ has no S-rooted model of K_t. Then, by the induction hypothesis, there exists $S' \subseteq V(C)$ with $S \subseteq S'$ such that $\mathrm{torso}_{G'}(S')$ is K_{2t}-minor-free. Since $\mathrm{torso}_{G'}(S') = \mathrm{torso}_G(S')$, this proves the theorem. □

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Structures des classes de graphes et de leurs mineurs exclus

Clément RAMBAUD

Résumé

Une classe de graphes est dite close par mineur si elle est close par suppressions d'arêtes, suppressions de sommets, et contractions d'arêtes. Les classes de graphes closes par mineur jouent un rôle central en théorie des graphes grâce à leurs propriétés structurelles et algorithmiques. Dans cette thèse, nous démontrons plusieurs correspondances entre la structure d'une classe de graphes close par mineur et celle de ses mineurs exclus, c'est-à-dire des graphes minimaux parmi ceux qui ne sont pas membres de cette classe.

Dans une première partie, nous démontrons une propriété structurelle pour les classes de graphes excluant une grille de hauteur fixée en tant que mineur. Pour ce faire, nous introduisons une nouvelle famille de paramètres de graphes qui généralise la profondeur arborescente et la largeur arborescente. En conséquence, nous obtenons une généralisation du Théorème de la Grille Mineure de Robertson et Seymour.

Dans une seconde partie, nous montrons, à travers plusieurs applications, comment utiliser une notion de mineurs enracinés pour résoudre des problèmes sur les mineurs de graphes. La première de ces applications est une preuve simple pour les caractérisations des classes de graphes closes par mineurs ayant une profondeur arborescente en couche ou une largeur linéaire en couche bornée. Une deuxième application consiste en des théorèmes de Structure Produit dans des classes closes par mineurs. Enfin, nous déterminons, à un facteur linéaire près, les nombres chromatiques centrés ainsi que les nombres colorant faibles de toute classe de graphes close par mineur donnée. Dans le cas où cette classe exclut un graphe planaire, nos bornes sont optimales à un facteur constant près.

Mots-clés : théorie des graphes, mineurs de graphes, largeur arborescente, profondeur arborescente, colorations centrées

Abstract

A class of graphs is said to be minor-closed if it is closed under the following three operations: edge deletion, vertex deletion, and edge contraction. Minor-closed classes of graphs play a central role in graph theory thanks to their numerous structural and algorithmic properties. In this thesis, we prove several connections between the structure of a minor-closed class of graphs and the structure of its excluded minors, that is the minimal graphs which are not members of this class.

In a first part, we show a structural property for classes of graphs excluding a grid of fixed height as a minor. To do so, we introduce a new family of graph parameters generalizing both treedepth and treewidth. As a consequence, we obtain a qualitative strengthening of the Grid-Minor Theorem of Robertson and Seymour for graphs excluding a rectangular grid.

In a second part, we show through multiple applications how to use a notion of rooted minors to solve problems concerning graph minors. As a first application, we provide simple proofs for characterizations of minor-closed classes of graphs having bounded layered treedepth or layered pathwidth. A second application consists of Product Structure theorems in minor-closed classes of graphs. Finally, we investigate the growth rates in minor-closed classes of graphs of weak coloring numbers and centered chromatic numbers, two families of graphs parameters characterizing classes of graphs having bounded expansions. In particular, we determine, up to a linear factor, the maximum centered chromatic numbers and weak coloring numbers of the members of a given minor-closed class of graphs. In the special case where a planar graph is excluded, our bounds are tight up to a constant factor.

Keywords: graph theory, graph minors, treewidth, treedepth, centered colorings