Structures of graph classes and of their excluded minors

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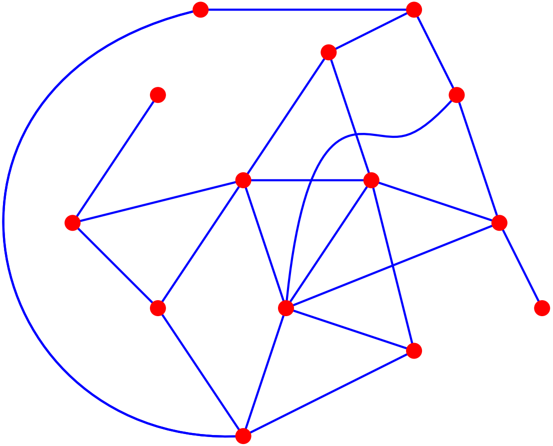
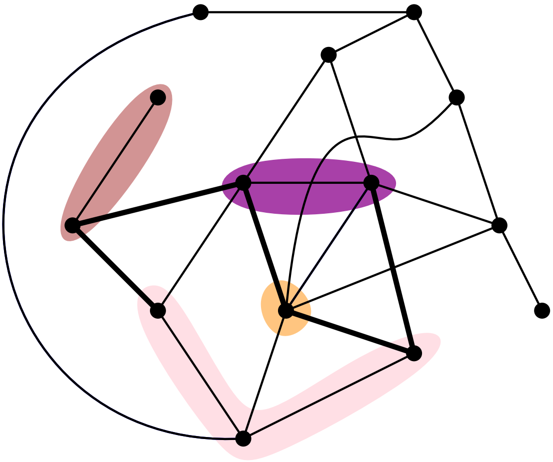
Graphs

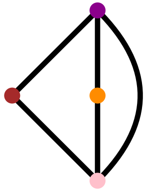
A **graph** is made of vertices and edges.



Graph minors and models

A **minor** is obtained by contracting *disjoint connected* subgraphs, and removing some vertices and edges.





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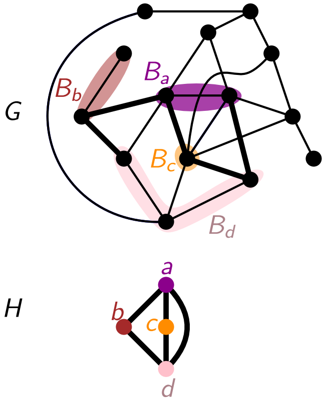


A family $(B_x \mid x \in V(H))$ is a **model** of H in G if

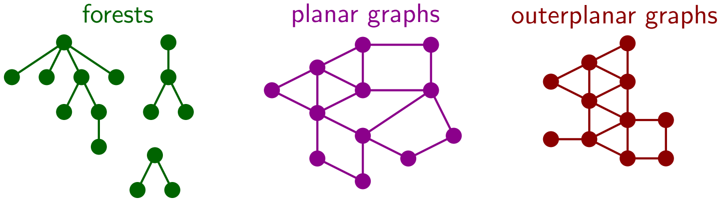
1. $G[B_x]$ for $x \in V(H)$ are disjoint and connected, and
2. there is an **edge** between B_x and B_y for every $xy \in E(H)$.

Minor-closed classes of graphs

$\mathcal{C}$ is **minor-closed** if

$$\forall G \in \mathcal{C}, \forall H \text{ minor of } G, H \in \mathcal{C}.$$

Examples:



Many usual classes of graphs are minor-closed.

$\rightarrow$ we want to understand their structure.

Minor-closed classes of graphs and excluded minors

Theorem (Robertson-Seymour Theorem)

Let $\mathcal{C}$ be a minor-closed class of graphs. There exists a finite list $X_1, \ldots, X_\ell$ of graphs such that

$$\mathcal{C} \text{ is the class of } \{X_1, \ldots, X_\ell\}\text{-minor-free graphs.}$$

I.e. $\forall G, \quad G \in \mathcal{C} \quad \Leftrightarrow \quad \forall i \in [\ell], X_i$ is not a minor of G.

Examples:

$$\begin{aligned} G \text{ forest} &\quad\Leftrightarrow\quad G\ \{K_3\}\text{-minor-free} \\ G \text{ planar} &\quad\Leftrightarrow\quad G\ \{K_5, K_{3,3}\}\text{-minor-free} \\ G \text{ outerplanar} &\quad\Leftrightarrow\quad G\ \{K_4, K_{2,3}\}\text{-minor-free} \end{aligned}$$

Minor-closed classes of graphs and excluded minors

Theorem (Robertson-Seymour Theorem)

Let $\mathcal{C}$ be a minor-closed class of graphs. There exists a finite list $X_1, \ldots, X_\ell$ of graphs such that

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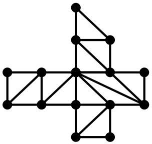
I.e. $\forall G, \quad G \in \mathcal{C} \quad \Leftrightarrow \quad \forall i \in [\ell], X_i$ is not a minor of G.

General question:

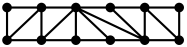
What are the links between the structure of $X_1, \ldots, X_\ell$ and the structure of $\{X_1, \ldots, X_\ell\}$-minor-free graphs?

Some tools of Graph Minor Theory

- tree decompositions and treewidth
  $\rightarrow$ Grid-Minor Theorem (Robertson and Seymour)



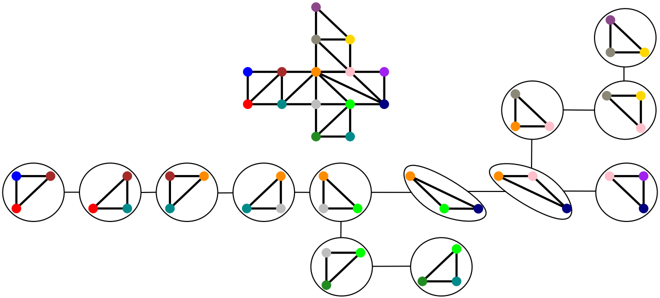
- path decompositions and pathwidth
  $\rightarrow$ Excluded-Tree-Minor Theorem (Robertson and Seymour)



Tree decompositions & treewidth

Tree decomposition: $\big(T, (W_x \mid x \in V(T))\big)$ such that

1. $\forall uv \in E(G), \exists x \in V(T)$ s.t. $u, v \in W_x$,
2. $T[\{x \in V(T) \mid u \in W_x\}]$ is *nonempty* and *connected*, $\forall u \in V(G)$.



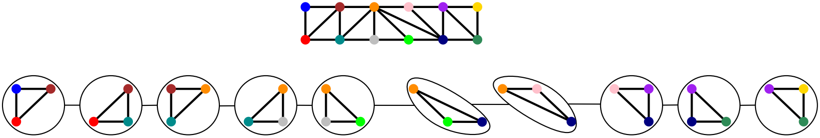
Width $= \max_{x \in V(T)} |W_x| - 1$.

Treewidth: $\mathrm{tw}(G) =$ minimum width of a tree decomposition.

Path decompositions & pathwidth

Path decomposition: $(W_1, \ldots, W_\ell)$ such that

1. $\forall uv \in E(G), \exists i \in [\ell]$ s.t. $u, v \in W_i$,
2. $\{i \in [\ell] \mid u \in W_i\}$ is a *nonempty interval*, $\forall u \in V(G)$.



Width $= \max_{1 \leqslant i \leqslant \ell} |W_i| - 1$.

Pathwidth: $\mathrm{pw}(G) =$ minimum width of a path decomposition.

Bag: set of the form W_i.

Adhesion: set of the form $W_i \cap W_{i+1}$.

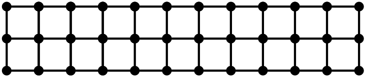
Contributions

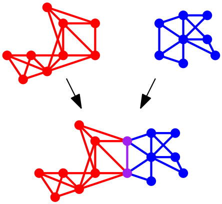
This thesis is based on:

- **Excluding a rectangular grid**
  Rambaud
- **Quickly excluding an apex-forest**
  Hodor, La, Micek, Rambaud; to appear in *SIDMA*
- **The grid-minor theorem revisited**
  Dujmović, Hickingbotham, Hodor, Joret, La, Micek, Morin, Rambaud, Wood; in *SODA 24* and *Combinatorica*
- **Weak coloring numbers of minor-closed graph classes**
  Hodor, La, Micek, Rambaud; in *SODA 25*
- **Centered colorings in minor-closed graph classes**
  Hodor, La, Micek, Rambaud; to appear in *SODA 26*

Part I

Excluding a rectangular grid

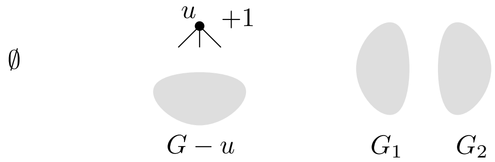




Treedepth

The **treedepth** is the *largest* parameter td satisfying

1. $\mathrm{td}(\emptyset) = 0$,
2. $\mathrm{td}(G) \leqslant 1 + \mathrm{td}(G - u)$, and
3. $\mathrm{td}(G_1 \sqcup G_2) \leqslant \max\{\mathrm{td}(G_1), \mathrm{td}(G_2)\}$.



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$$\mathrm{td} \leqslant 0$$

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$$\mathrm{td} \leqslant 1$$

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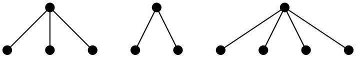
$\mathrm{td} \leqslant 2$



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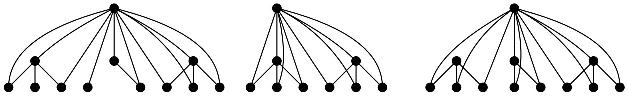
$\mathrm{td} \leqslant 3$



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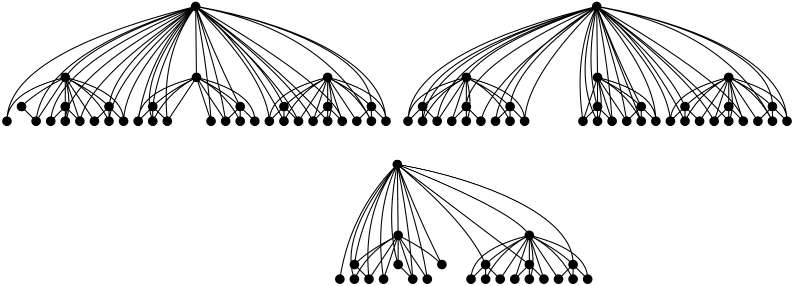
$\mathrm{td} \leqslant 4$



Treedepth of paths

Treedepth of paths

Property

Paths have unbounded treedepth:

$$\mathrm{td}(P_{2^k}) > k$$

Proof: by induction on k.

Consequence: a long path is a certificate of large treedepth.

Graphs with no long paths

Theorem (Nešetřil and Ossona de Mendez)

If G has no path of length ℓ, then

$$\mathrm{td}(G) \leqslant \ell.$$

Proof: a DFS.

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Theorem (Nešetřil and Ossona de Mendez)

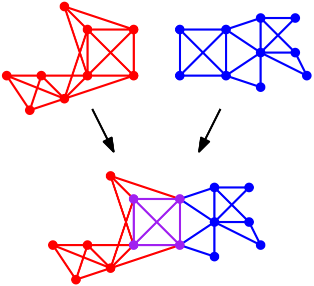
The following are equivalent.

1. $\mathcal{C}$ has bounded td, i.e. $\exists N, \forall G \in \mathcal{C}, \mathrm{td}(G) \leqslant N$,
2. there is an integer ℓ such that no graph in $\mathcal{C}$ contains P_ℓ as a subgraph/minor.

Treewidth

The **treewidth** is the *largest* parameter satisfying

1. $\mathrm{tw}(\emptyset) = -1$,
2. $\mathrm{tw}(G) \leqslant 1 + \mathrm{tw}(G - u)$, and
3. $\mathrm{tw}(G) \leqslant \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$ if G is a *clique-sum* of G_1 and G_2.



(and possibly removing some edges)

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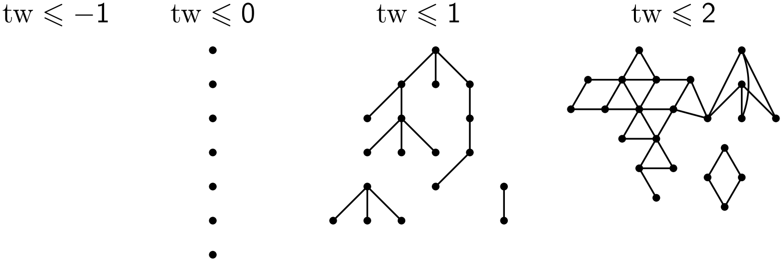


Treewidth and grids

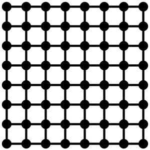
Property

If H is a minor of G, then

$$\mathrm{tw}(H) \leqslant \mathrm{tw}(G).$$

Property

Grids have unbounded treewidth.



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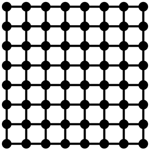

Theorem (Grid-Minor Theorem, Robertson and Seymour; 1986)

The following are equivalent.

1. $\mathcal{C}$ has bounded tw, i.e. $\exists N, \forall G \in \mathcal{C}, \mathrm{tw}(G) \leqslant N$,
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k-treedepth

The k-**treedepth** is the *largest* parameter satisfying

1. $\mathrm{td}_k(\emptyset) = 0$,
2. $\mathrm{td}_k(G) \leqslant 1 + \mathrm{td}_k(G - u)$, and
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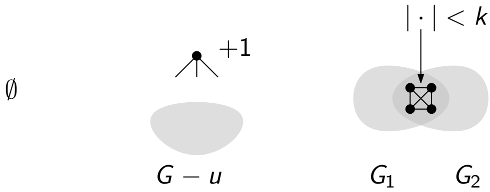

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Examples:

- $\mathrm{td}_1 = \mathrm{td}$
- $\mathrm{td}_2 = \mathrm{td}_2$ (Huynh, Joret, Micek, Seweryn, and Wollan; 2020)
- $\mathrm{td}_{+\infty} = 1 + \mathrm{tw}$

Obstructions for k-treedepth

Property

If H is a minor of G, then

$$\mathrm{td}_k(H) \leqslant \mathrm{td}_k(G).$$

Obstructions for k-treedepth

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If H is a minor of G, then

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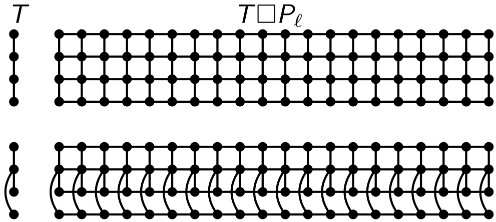
Property

For every tree T on k vertices, $\{T \square P_\ell\}_{\ell \geqslant 1}$ have unbounded td_k.



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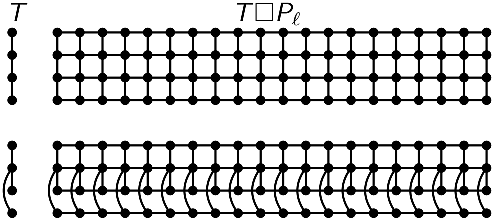

Theorem (Rambaud; 2025+)

The following are equivalent.

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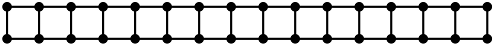
2. there is an integer ℓ such that for every tree T on k vertices, no graph in $\mathcal{C}$ contains $T \square P_\ell$ as a minor.

The obstructions

Obstructions for td_1

The obstructions

Obstructions for td_2

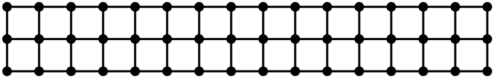


Theorem (Huynh, Joret, Micek, Seweryn, and Wollan; 2020)

A class of graphs has bounded td_2 iff it excludes a *ladder* as a minor.

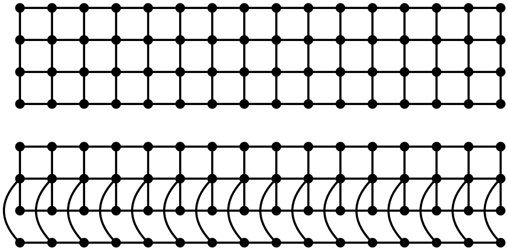
The obstructions

Obstructions for td_3



The obstructions

Obstructions for td_4



Excluding the $k \times \ell$ grid

Setting: k fixed.

Corollary (Rectangular Grid-Minor Theorem)

Graphs excluding the $k \times \ell$ grid as a minor have bounded td_{2k-1}.

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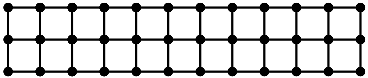
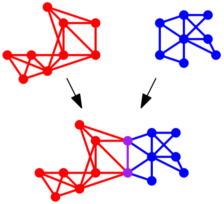
height of the excluded grid $\leftrightarrow$ size of the clique-sums



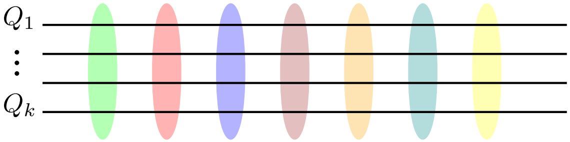


A glimpse of the proof: a key lemma

Theorem (Rambaud; 2025+)

If, for every tree T on k vertices, $T \square P_\ell$ is not a minor of G, then

$$\mathrm{td}_k(G) \leqslant f(k, \ell).$$



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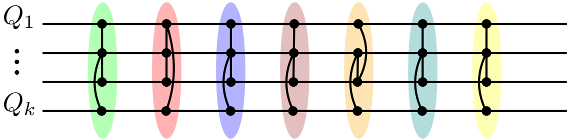

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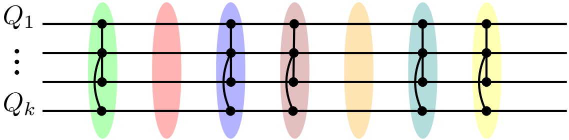
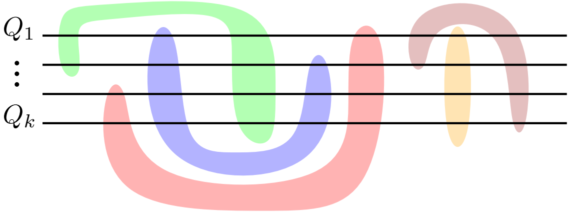


A glimpse of the proof: a key lemma

Lemma

Let $Q_1, \dots, Q_k$ be k disjoint paths. If there are $f(k, \ell)$ pairwise disjoint connected subgraphs each intersecting every Q_i, then G contains a $T \square P_\ell$ as a minor, for some tree T on k vertices.



A glimpse of the proof: a key lemma

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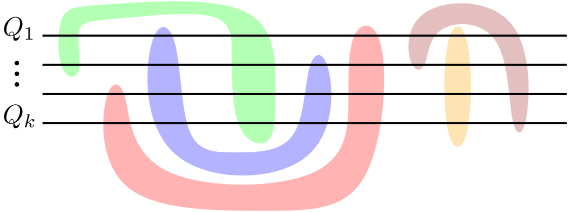


Lemma (Folklore)

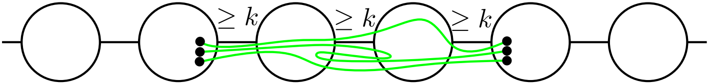
For every family $\mathcal{F}$ of connected subgraphs, if there are no $d+1$ disjoint members of $\mathcal{F}$, then there is a hitting set of size at most $d \cdot (\mathrm{tw}(G)+1)$.

A glimpse of the proof: the bounded pw case

Theorem (Robertson and Seymour, unpublished)

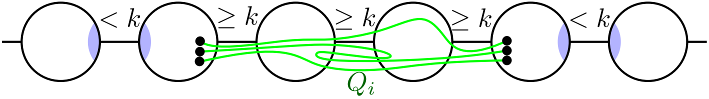
There is a path decomposition $(W_1, \ldots, W_\ell)$ of width $\mathrm{pw}(G)$ such that for every $1 \leqslant x < y \leqslant \ell$, for every $k \geqslant 0$,

1. there exists $z \in \{x, \ldots, y-1\}$ such that $|W_z \cap W_{z+1}| < k$, or
2. there are k disjoint (W_x, W_y)-paths in G.



A glimpse of the proof: the bounded pw case

Assumptions: no $T \Box P_\ell$ minor for every tree T on k vertices.
Goal: $\mathrm{td}_k(G) \leqslant f(k, \ell, \mathrm{pw}(G))$.



Not too many disjoint connected subgraphs intersecting every Q_i.

A glimpse of the proof: the bounded pw case

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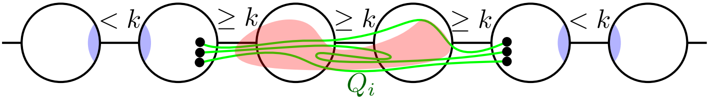
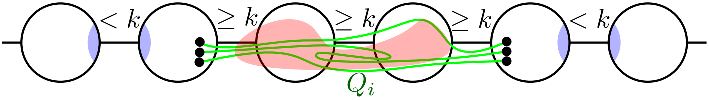


Not too many disjoint connected subgraphs intersecting every Q_i
$\Rightarrow$ small hitting set.

A glimpse of the proof: the bounded pw case

Assumptions: no $T \square P_\ell$ minor for every tree T on k vertices.
Goal: $\mathrm{td}_k(G) \leqslant f(k, \ell, \mathrm{pw}(G))$.



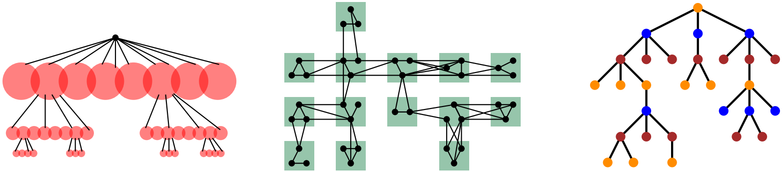
Not too many disjoint connected subgraphs intersecting every Q_i
$\Rightarrow$ small hitting set.

$\rightarrow$ induction on the components of what remains.

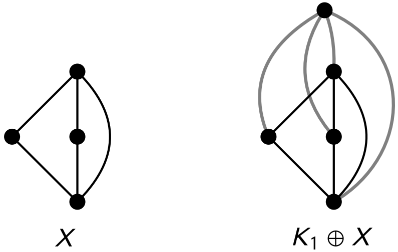
$$\mathrm{td}_k(G) \leqslant 2(k-1) + f(k, \ell)(\mathrm{tw}(G) + 1) + \mathrm{induction}(\mathrm{pw}(G) - 1)$$

Part II

Rooted minors and applications



Adding an apex



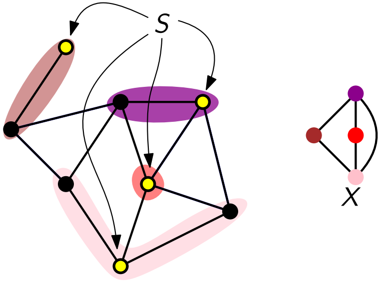
Question:

How to deduce a structure for $(K_1 \oplus X)$-minor-free graphs, knowing a structure on X-minor-free graphs ?

Rooted models

Let $S \subseteq V(G)$. A model $(B_x \mid x \in V(X))$ of X is S-**rooted** if

$$B_x \cap S \neq \emptyset \text{ for every } x \in V(X).$$



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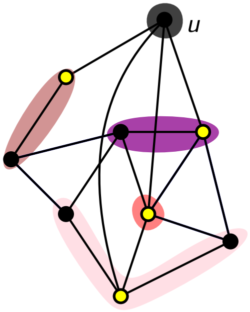
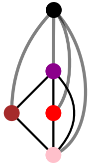




Observation

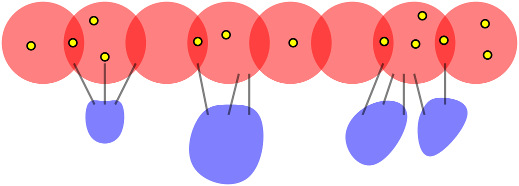
$N(u)$-rooted model of X in $G - u \Rightarrow$ model of $K_1 \oplus X$ in G.

Application: layered pathwidth

Setting: $S \subseteq V(G)$, X a forest

Theorem (Hodor, La, Micek, Rambaud; 2024+)

No S-rooted model of X
$\Rightarrow$ *pathwidth "focused on S"* at most $2|V(X)| - 2$.



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Corollary

No model of $K_1 \oplus X \Rightarrow$ *layered pathwidth* at most $2|V(X)| - 1$.

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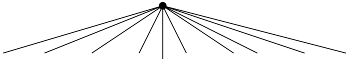

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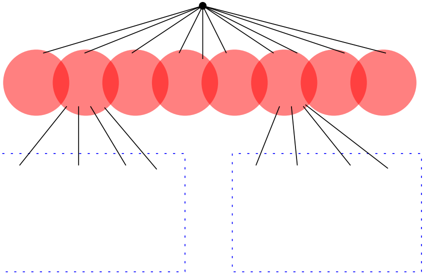

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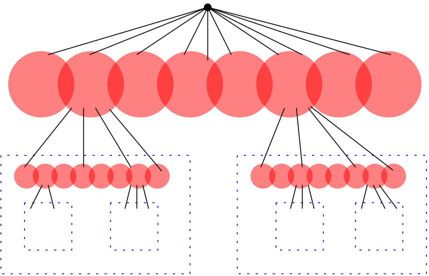

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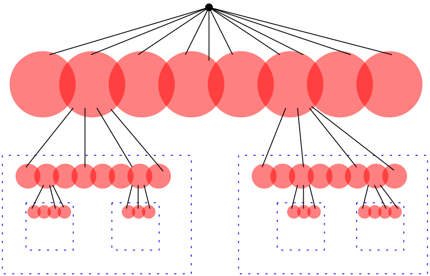

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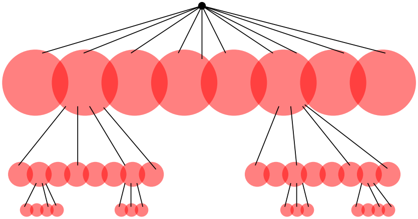
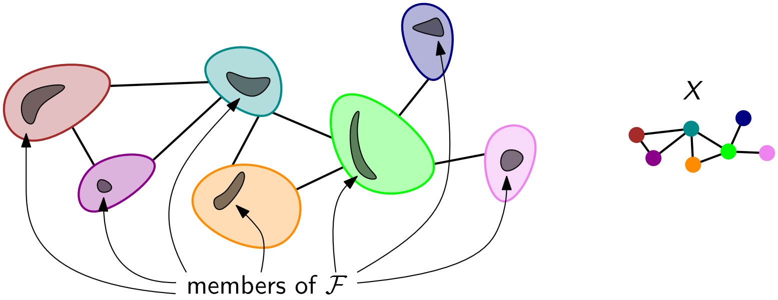


More than adding one vertex: rich models

Let $\mathcal{F}$ be a family of connected subgraphs of G.
A model $(B_x \mid x \in V(X))$ of X is $\mathcal{F}$-**rich** if

$$\forall x \in V(X), \exists F \in \mathcal{F}, F \subseteq G[B_x].$$



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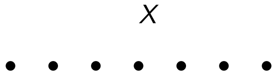
We are now looking for properties of the form:

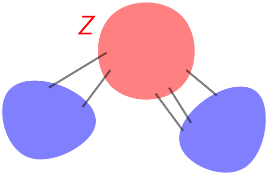
No $\mathcal{F}$-rich model of $X \Rightarrow$ well-structured hitting set Z for $\mathcal{F}$.

$\rightarrow$ This allows to set up inductions on the excluded minor.

Rich models *vs* hitting sets

Assuming $\mathrm{tw}(G)$ bounded:





no $\mathcal{F}$-rich model of $X \Rightarrow$ hitting set Z of $\mathcal{F}$ with $|Z|$ bounded.

Rich models *vs* hitting sets

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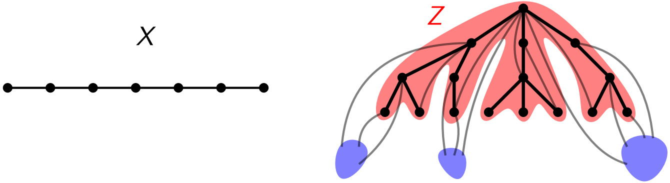

no $\mathcal{F}$-rich model of $X \Rightarrow$ hitting set Z of $\mathcal{F}$ with $\mathrm{td}(G, Z)$ bounded.

Rich models *vs* hitting sets

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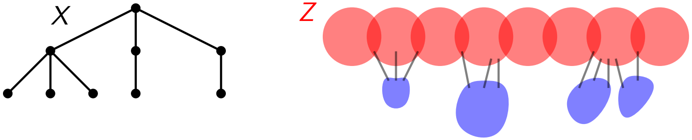


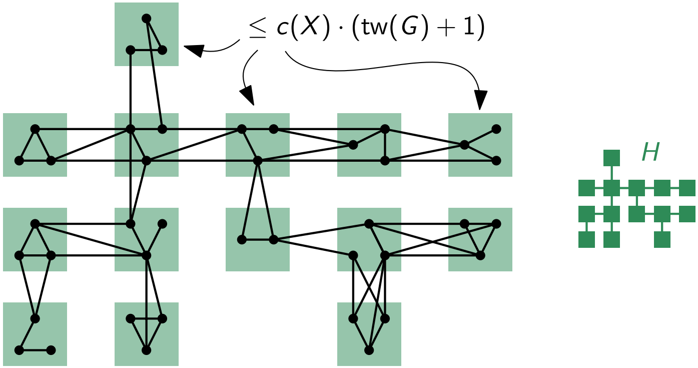
no $\mathcal{F}$-rich model of $X \Rightarrow$ hitting set Z of $\mathcal{F}$ with $\mathrm{pw}(G, Z)$ bounded.

Application: product structure

Theorem (DHHJLMMRW; 2024+)

For every X-minor-free graph G, there exists a graph H such that

1. $\mathrm{tw}(H) \leqslant 2^{\mathrm{td}(X)} - 2$, and
2. $G \subseteq H \boxtimes K_{c(X)\cdot(\mathrm{tw}(G)+1)}$.



Application: centered colorings

$\varphi\colon V(G) \to C$ is q-**centered** if for every connected subgraph H of G, either

1. $|\varphi(V(H))| > q$, or
2. there is a color $c \in C$ that appears exactly once in $V(H)$.

Notation: $\mathrm{cen}_q(G) =$ min # of colors in a q-centered coloring.

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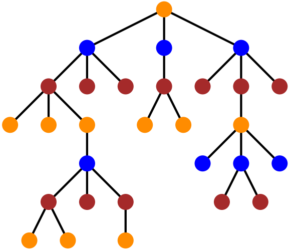
$q = 2$



Property

Trees have q-centered colorings using $q + 1$ colors.

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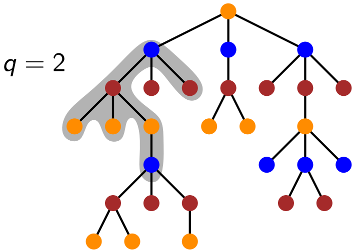

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Theorem (Mi. Pilipczuk and Siebertz; 2019)

K_t-minor-free graphs have $\mathrm{cen}_q(\cdot) \leqslant \mathcal{O}(q^{f(t)})$.

Theorem (Dębski, Felsner, Micek, and Schröder; 2021)

There are K_t-minor-free graphs with $\mathrm{cen}_q(\cdot) \geqslant \Omega(q^{t-2})$.

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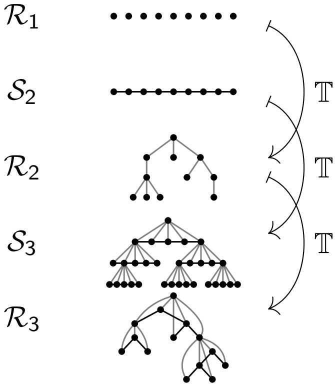
Given $X_1, \ldots, X_\ell$, one can determine

$$\max \left\{ \mathrm{cen}_q(G) \;\middle|\; G \ \{X_1, \ldots, X_\ell\}\text{-minor-free} \right\}$$

up to a $\mathcal{O}(q)$-factor.

Centered colorings: known lower bounds

$$\mathcal{R}_1 = \{\text{edgeless graphs}\}$$
$$\mathcal{R}_{t+1} = \mathbb{T}(\mathcal{R}_t)$$
$$\mathcal{S}_2 = \{\text{linear forests}\}$$
$$\mathcal{S}_{t+1} = \mathbb{T}(\mathcal{S}_t)$$



Theorem (Dębski, Felsner, Micek, and Schröder; 2021)

$$\max_{G \in \mathcal{R}_t} \operatorname{cen}_q(G) \geqslant \Omega(q^{t-1})$$

$$\max_{G \in \mathcal{S}_t} \operatorname{cen}_q(G) \geqslant \Omega(q^{t-2} \log q)$$

Generic bounds

$\rightarrow$ Up to a $\mathcal{O}(q)$-factor, that's the only constructions.

Theorem (Hodor, La, Micek, Rambaud; 2025+)

Let $t \geqslant 3$ and let $\mathcal{C}$ be a minor-closed class of graphs.

1. If $\mathcal{C}$ excludes a member of $\mathcal{R}_t$, then

$$\max_{G \in \mathcal{C}} \mathrm{cen}_q(G) \leqslant \mathcal{O}(q^{t-1} \log q).$$

2. If $\mathcal{C}$ excludes a member of $\mathcal{S}_t$, then

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Always gives a bound tight up to a $\mathcal{O}(q)$-factor.

Generic bounds — bounded treewidth

For **bounded treewidth** graphs,

→ Up to a $\mathcal{O}(1)$-factor, that's the only constructions.

Theorem (Hodor, La, Micek, Rambaud; 2025+)

Let $t \geqslant 3$ and let $\mathcal{C}$ be a minor-closed class of graphs having **bounded treewidth**.

1. If $\mathcal{C}$ excludes a member of $\mathcal{R}_t$, then

$$\max_{G \in \mathcal{C}} \mathrm{cen}_q(G) \leqslant \mathcal{O}(q^{t-2} \log q).$$

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Always gives a bound tight up to a $\mathcal{O}(1)$-factor.

Other similar bounds

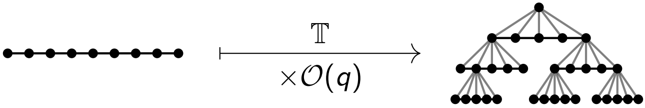
We obtain similar bounds for

- *weak coloring numbers,*
- *fractional* td*-fragility rates,*

$\{\mathrm{cen}, \mathrm{wcol}, \mathrm{ftdfr}\} \times \{\mathrm{tw} < +\infty, \mathrm{tw} = +\infty\} \times \{\mathcal{R}_t, \mathcal{S}_t\} =$ 12 bounds.

Our approach: find the correct "$\mathcal{F}$-rich/$\mathcal{F}$-hitting-set" statement

1. separate base cases,
2. one common induction step $\mathcal{X} \mapsto \mathbb{T}(\mathcal{X})$.



Conclusion and open problems

Problem

Find other applications of rooted/rich models.
What about topological minors ?

Conclusion and open problems

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Find other applications of rooted/rich models.
What about topological minors ?

Conjecture (Thomas; 1989)

Every minor-monotone graph parameter admits finitely many classes of obstructions.

Examples:

$$\mathrm{tw} \leftrightarrow \{\text{planar graphs}\} \qquad \text{(Robertson and Seymour)}$$

$$\mathrm{pw} \leftrightarrow \{\text{forests}\} \qquad \text{(Robertson and Seymour)}$$

$$\mathrm{td}_k \leftrightarrow \{\{T \square P_\ell \mid \ell\} \mid |V(T)| = k\}$$

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Thank you !